Baryon wave function: Large-N_c QCD and lessons from models

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The structure of the $1/N_c$ expansion for the baryon distribution amplitude in QCD is tested using quark models. Earlier conjectures about this structure based on the evolution equation and on the soft-pion theorem are confirmed by the model analysis. The problem of the calculation of the baryon wave function at large N_c is reduced to the analysis of equations of motion for an effective classical dynamical system.

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I. INTRODUCTION

A. Large N_c: QCD and models

The limit of the large number of colors N_c and the $1/N_c$ expansion [1]–[5] are important nonperturbative methods in QCD. However, in spite of the long studies of the large-N_c limit, QCD is not solved even in the leading order of the $1/N_c$ expansion. In the applications of the $1/N_c$ expansion to QCD, one usually deals not with the order-by-order *calculation* of the $1/N_c$ corrections but with the analysis of the *structure* of the $1/N_c$ expansion.

The failure of the dynamical approach to the $1/N_c$ expansion in QCD led to the redistribution of the effort towards various models imitating large-N_c QCD. This approach proved to be rather fruitful. Instead of being trapped into the study of model artifacts (which often happens in the phenomenology of strong interactions), the analysis of various large-N_c models has revealed a number of their common properties including the so-called large-N_c spin-flavor symmetry [6]–[11] which is believed to be a symmetry of large-N_c QCD as well.

A classical example of results, historically first derived in the Skyrme model but actually based on the spin-flavor symmetry of large-N_c QCD, is the ratio of the pion-nucleon ($g_{\pi NN}$) and pion-nucleon-Δ ($g_{\pi N\Delta}$) constants [12]

$$\frac{g_{\pi N\Delta}}{g_{\pi NN}} = \frac{3}{2}\left[1 + O(N_c^{-2})\right] \quad (\text{exp. } 1.48)\,. \tag{1.1}$$

The same ratio 3/2 can be derived in the quark model [13] in the limit $N_c \to \infty$.

This simple example shows that the study of large-N_c models can be helpful for establishing the properties of large-N_c QCD itself.

B. Baryon wave function in the large-N_c limit

This paper is devoted to the $1/N_c$ expansion for the baryon wave function. This problem was raised in Refs. [14, 15]. The construction of the $1/N_c$ expansion for the baryon wave function is rather nontrivial, since one has to deal with the exponential large-N_c behavior. In Refs. [16, 17] it was shown that this exponential behavior can be consistently described using a specially designed generating functional for the baryon wave function. Strictly speaking, the results of Refs. [16, 17] are based on *conjectures*, whose consistency was checked using various indirect methods like evolution equation [16] and soft-pion theorem [17]. In this paper we provide additional arguments supporting these conjectures, using models of large-N_c QCD.

Before describing the main features of the $1/N_c$ expansion for the baryon wave function we would like to agree about the definition of the baryon wave function. The concept of the wave function is natural for nonrelativistic quantum mechanics but its incorporation in QCD requires some extra specifications. Usually one keeps in mind the transition matrix element between baryon $|B\rangle$ and vacuum $|0\rangle$

$$\langle 0| \prod_{c=1}^{N_c} \psi_c(z_c) |B\rangle\,, \tag{1.2}$$

where $\psi_c(z)$ is the quark field with the color index c. Strictly speaking one has to settle several questions:

1) ordering of quark operators $\psi_c(z_c)$,
2) insertion of Wilson lines providing the gauge invariance,
3) choice of the space-time or momentum kinematics (equal-time wave function, light-cone wave function etc.).

From the point of view of QCD applications to hard exclusive processes the most interesting object is the *baryon distribution amplitude* [18, 19, 20] which corresponds to the choice

$$z_c = \lambda_c n, \quad n^2 = 0 \tag{1.3}$$

in Eq. (1.2). Here n is a fixed light-cone vector. The standard definition of the baryon distribution amplitude also involves the Fourier transformation from parameters λ_c to variables x_c corresponding to the momentum fractions of quarks in the infinite momentum frame:

$$\int d\lambda e^{i\lambda_k x_k} \langle 0| \prod_{k=1}^{N_c} \psi_{c_k}(\lambda_k n)|B\rangle \sim \varepsilon_{c_1 c_2 \ldots c_{N_c}} \Psi_B(x_1, x_2, \ldots, x_{N_c})\,. \tag{1.4}$$

We have omitted various normalization factors which are not important for the discussion of the general properties of the $1/N_c$ expansion.

Although the light-cone elements of the definition of the baryon distribution amplitude are important from the *dynamical* point of view, they are rather insignificant for the *structure of the* $1/N_c$ *expansion*. What is really important for the construction of the $1/N_c$ expansion is that the baryon wave function depends on N_c quark variables. At this moment one meets a serious problem: how can one construct the $1/N_c$ expansion for the function (1.4) which depends on N_c variables x_k? This problem is much more general than the context of the baryon distribution amplitude. Below we will consider the general case without specifying the precise meaning of the wave function $\Psi_B(x_1, x_2, \ldots, x_{N_c})$ and of its arguments $x_1, x_2, \ldots, x_{N_c}$. For example, the variables x_k can be composite objects carrying information about the spin and flavor of quarks in addition to their coordinates or momenta. Since the antisymmetric color tensor $\varepsilon_{c_1 c_2 \ldots c_{N_c}}$ is factored out in Eq. (1.4), the function $\Psi_B(x_1, x_2, \ldots, x_{N_c})$ is symmetric in all its arguments $x_1, x_2, \ldots, x_{N_c}$.

In Refs. [16, 17] the problem of the construction the $1/N_c$ expansion for function depending on N_c variables was solved by introducing a generating functional for the baryon wave function. Omitting irrelevant technical details (spin and flavor structure, light-cone kinematics, $1/N_c$ rescaling of the quark momentum, etc.) and concentrating on the general structure which is important for the large-N_c limit, we can define this generating functional as

$$\Phi_B(g) = \int \Psi_B(x_1, x_2, \ldots, x_{N_c}) g(x_1) g(x_2) \ldots g(x_{N_c}) dx_1 dx_2 \ldots dx_{N_c}\,, \tag{1.5}$$

where $g(x)$ is an arbitrary "source" function. Since function $\Psi_B(x_1, x_2, \ldots, x_{N_c})$ is symmetric in permutations of $x_1, x_2, \ldots, x_{N_c}$, the transition from the wave function $\Psi_B(x_1, x_2, \ldots, x_{N_c})$ to the functional $\Phi_B(g)$ does not lead to any loss of information at finite N_c.

As was argued in Refs. [16, 17], the functional $\Phi_B(g)$ has an exponential behavior at large N_c:

$$\Phi_B(g) \overset{N_c\to\infty}{\sim} \exp\left[N_c W(g)\right]\,. \tag{1.6}$$

where $W(g)$ is some (N_c independent) functional of g. An important property of the functional $W(g)$ is its universality: all low-lying baryons [with $O(N_c^{-1})$ or $O(N_c^0)$ excitation energy] are described by the same functional $W(g)$.

The dependence on the type of the baryon B appears in the preexponential factor $A_B(g)$:

$$\Phi_B(g) \overset{N_c\to\infty}{=} N_c^{\nu_B} A_B(g) \exp\left[N_c W(g)\right] \left[1 + O(N_c^{-1})\right]. \tag{1.7}$$

Although the functional $A_B(g)$ depends on the baryon B, this dependence has simple factorization properties studied in Ref. [17].

Strictly speaking, these properties of functionals $W(g)$ and $A_B(g)$ were not properly derived in Refs. [16, 17]. Instead the consistency of these properties with a number of basic features of QCD was checked:
– QCD evolution equation [16],
– asymptotic behavior in terms of anomalous dimensions of leading twist operators [16],
– spin-flavor symmetry of large-N_c QCD [17],
– soft-pion theorem for the baryon distribution amplitude [17].

Although all these consistency checks successfully passed, they cannot be considered as a final proof. In this paper we turn to the analysis of large-N_c models in order to find additional arguments supporting the large-N_c behavior (1.7) and illustrating the basic properties of functionals $W(g)$ and $A_B(g)$.

C. Consequences of the spin-flavor symmetry

In the absence of the dynamical solution for large-N_c QCD, one cannot expect to find much more than relations like the N-Δ ratio (1.1) in the pure large-N_c approach. In Ref. [17] a relation was derived for the generating functionals $\Phi^{(N)}_{T_3J_3}(g)$ and $\Phi^{(\Delta)}_{T_3J_3}(g)$ describing the baryon wave functions of nucleon and Delta resonance where T_3 and J_3 are projections of isospin and spin, respectively.

In order to write this relation let us first notice that $\Phi^{(N)}_{T_3J_3}(g)$ is a 2×2 matrix with indices T_3, J_3. Let us consider the 2×2 matrix

$$R = \frac{\Phi^{(N)}(g)}{\sqrt{\det \Phi^{(N)}(g)}}\,. \tag{1.8}$$

Obviously

$$\det R = 1 \tag{1.9}$$

so that matrix R belong to the group $SL(2,C)$. The irreducible representations of $SL(2,C)$ are described by Wigner functions $D^{(j_1,j_2)}(R)$ depending on two "spins" j_1, j_2. Let us consider the Wigner function corresponding to $j_1 = 3/2$, $j_2 = 0$. For brevity we will denote it simply $D^{3/2}_{T_3J_3}$. Now we are ready to write the N-Δ relation derived in Ref. [17]:

$$\Phi^{(\Delta)}_{T_3J_3}(g) = D^{3/2}_{T_3J_3}\left[\frac{\Phi^{(N)}(g)}{\sqrt{\det \Phi^{(N)}(g)}}\right]\sqrt{2\det \Phi^{(N)}(g)}\left[1+O(N_c^{-1})\right]\,. \tag{1.10}$$

Relation (1.10) was derived in Ref. [17] using the realization of the large-N_c spin-flavor symmetry in the space of the preexponential functionals $A_B(g)$ appearing in Eq. (1.7).

Due to the universality of the functional $W(g)$, relation (1.10) is equivalent to the analogous relation for the preexponential functionals $A^{(\Delta)}_{T_3J_3}(g)$ and $A^{(N)}_{T_3J_3}(g)$ appearing in Eq. (1.7):

$$A^{(\Delta)}_{T_3J_3}(g) = D^{3/2}_{T_3J_3}\left[\frac{A^{(N)}(g)}{\sqrt{\det A^{(N)}(g)}}\right]\sqrt{2\det A^{(N)}(g)}\left[1+O(N_c^{-1})\right]\,. \tag{1.11}$$

D. What do we want of models?

It should be stressed that the construction of viable realistic models of the baryon distribution amplitude is not the aim of this paper. We are interested in another problem. We want to trace how the exponential large-N_c behavior (1.7) appears within a certain class of models. The main problem of large-N_c QCD is that its dynamics is not solved. Therefore the analysis of Refs. [16, 17] was based on the *assumption* about the exponential large-N_c behavior (1.7) with the universal functional $W(g)$.

In QCD we cannot compute the functional $W(g)$. On the contrary, in the models studied in this paper both the exponential behavior (1.7) and the calculation of $W(g)$ are under good theoretical control. From the point of view of the aim of this paper (model check of the assumptions made in large-N_c QCD), the result of this work is trivial: all assumptions involved in the theoretical construction [16, 17] are confirmed in the models studied in this paper. This puts the results of Refs. [16, 17] on more solid ground.

Several comments must be made about the status of the spin-flavor symmetry in our model analysis. Most of this work is based on the Hamiltonian (Schrödinger) analysis of models. As is well known, the mass difference $M_\Delta - M_N$ between the Δ resonance and nucleon is $O(N_c^{-1})$, whereas the masses M_N and M_Δ grow as $O(N_c)$. Therefore the mass splitting $M_\Delta - M_N$ is an effect of the next-to-next-to-leading order of the $1/N_c$ expansion. In this paper we do not go so deeply into the $1/N_c$ expansion. Actually the main part is devoted to the calculation of the functional $W(g)$ corresponding to the leading order. Therefore the subtleties of the realization of the spin-flavor symmetry are not important for the most of our analysis. Moreover, in our analysis we make no special assumptions about the symmetries of the Hamiltonian [only the $SU(N_c)$ color symmetry is important for us].

On the other hand, the readers interested in model implementations of the spin-flavor symmetry can be addressed to Ref. [16] where functionals $W(g)$ and $A_B(g)$ are computed in the naive quark model and the results explicitly agree with all constraints imposed by the spin-flavor symmetry.

E. Mean field approximation and large-N_c limit

In contrast to "unsolvable" large-N_c QCD, many large-N_c models allow for a straightforward construction of the $1/N_c$ expansion. Although the precise form of the $1/N_c$ expansion can be model dependent, many features are common since they are based on the mean field approximation which is justified in these models (but not in QCD) in the large-N_c limit. The manifestations of the underlying mean field approximation can be different: saddle point approximation in the path integral approach, the Hartree equation in the Hamiltonian approach to the models with explicit quark degrees of freedom, classical equations of motion in models based on meson fields, etc.

In this paper we work with models explicitly containing quark degrees of freedom. In the leading order of the $1/N_c$ expansion the solution of these models is described by the Hartree equation.

F. Density matrix or wave function?

This paper deals with the $1/N_c$ expansion for the functional $\Phi_B(g)$ (1.5) describing the baryon wave function. One can wonder how this approach is related to the mean field method. Note that the mean-field approach to fermion systems is based on equations for the *density matrix* whereas the functional $\Phi_B(g)$ is constructed in terms of the *wave function* $\Psi_B(x_1, x_2, \ldots, x_{N_c})$. Strictly speaking the transition from the multiparticle wave function to the single-particle density matrix

$$\rho(x_1', x_1) = \int dx_2 \ldots dx_{N_c} \Psi_B(x_1', x_2, \ldots, x_{N_c}) \Psi_B^*(x_1, x_2, \ldots, x_{N_c}) \tag{1.12}$$

leads to a *loss* of information. Therefore the knowledge of the mean field solution in terms of the density matrix $\rho(x_1', x_1)$ is not sufficient for the calculation of the functionals $W(g)$ and $A_B(g)$ describing the large-N_c behavior (1.7) of the baryon wave function. Nevertheless one can derive a closed equation for the functional $W(g)$ [See Sec. IV and Eq. (4.6)]. An essential part of this paper is devoted to the analysis of this equation and to the construction of its solutions.

In the leading order of the $1/N_c$ expansion the density matrix $\rho(x', x)$ is described by the Hartree equation which leads to the representation for $\rho(x', x)$ in terms of single particle wave functions $\psi_n(x)$ of occupied states:

$$\rho(x', x) \stackrel{N_c\to\infty}{\longrightarrow} \sum_{n:\text{occupied}} \psi_n(x')\psi_n^*(x)\,. \tag{1.13}$$

In the simplest models with only one "valence" level ψ_0 occupied by N_c quarks, the density matrix factorizes into the product

$$\rho(x', x) \stackrel{N_c\to\infty}{\longrightarrow} \psi_0(x')\psi_0^*(x)\,. \tag{1.14}$$

Here we assume that the trivial color term $\delta_{cc'}$ is factored out.

In the large-N_c limit the factorization of the density matrix (1.14) becomes asymptotically exact. Naively one could think that the wave function $\Psi(x_1, x_2, \ldots, x_{N_c})$ has a similar factorization at large N_c:

$$\Psi(x_1, x_2, \ldots, x_{N_c}) \stackrel{N_c\to\infty}{\longrightarrow} \prod_{k=1}^{N_c} \psi_0(x_k) \quad \text{(wrong!)}\,. \tag{1.15}$$

Unfortunately this statement is wrong. In order to see the violation of Eq. (1.15) let us multiply it by $\prod_{k=1}^{N_c} g(x_k)$ and integrate over all x_k. If Eq. (1.15) were correct, then we would obtain using Eq. (1.7)

$$W(g) = \ln\left[\int \psi_0(x) g(x) dx\right] \quad \text{(wrong!)}\,. \tag{1.16}$$

In Sec. VII we describe a model for which the functional $W(g)$ can be computed analytically [Eq. (7.74)]. This result explicitly shows the breakdown of the naive expectation (1.16).

Although the naive statement (1.15) about the factorization of the wave function can be often met in literature, this statement can be trusted only in the sense of the factorization of the density matrix (1.14) in the leading order of the $1/N_c$ expansion.

To summarize, we cannot use Eq. (1.16) for the calculation of the functional $W(g)$. Moreover, the knowledge of the solution of the Hartree equation is not sufficient for the calculation of $W(g)$. In this paper we derive a differential equation for $W(g)$ and construct its solution.

G. Factorization of the preexponential terms

The universality of the large-N_c behavior (1.6) holds only with the exponential accuracy. The preexponential functional $A_B(g)$ in Eq. (1.7) depends on the baryon (baryon-meson) state B. In Refs. [16, 17] several important factorization properties of these functionals were suggested and checked using the evolution equation, the spin-flavor symmetry and the soft-pion theorem. In this paper we verify these factorization properties by explicit calculations in large-N_c models.

One should keep in mind that the factorization properties of functionals $A_B(g)$ are determined by two ingredients:

1) the breakdown of the spin and flavor symmetries by the mean field solution and the restoration of these symmetries via the standard method of "the quantization of collective coordinates",

2) the spectrum of the harmonic fluctuations above the mean field solution.

If the mean field solution for the baryon had the same symmetries as the vacuum mean field solution, then we would have a simple oscillator-like spectrum of baryons

$$\mathcal{E}_B \equiv \mathcal{E}_{\{n_k\}} = N_c E_0 + \left(\Delta E_0 + \sum_k n_k \Omega_k \right) + O(N_c^{-1}) \,, \tag{1.17}$$

where the baryon excitations B are labeled by sets of integer numbers $\{n_k\}$ associated with elementary Ω_k excitations. In this case the factorization of $A_B(g)$ is described by the formula

$$A_B\,(g) \equiv A_{\{n_k\}}(g) = A^{(0)}(g) \prod_k \left[A_k(g)\right]^{n_k} \ . \tag{1.18}$$

In the case when the mean field solution breaks some symmetries (which are nevertheless restored by the collective quantization) the situation is more subtle and the precise expression for $A_B\,(g)$ depends on the involved symmetries. As is well known, the effects of broken symmetries in the mean field approach to large-N_c models correspond to the manifestation of the spin-flavor symmetry in large-N_c QCD. The structure of the functional $A_B\,(g)$ in large-N_c QCD and the role of the spin-flavor symmetry were studied in Ref. [17].

In this paper we concentrate on models where the effects of broken symmetries and zero modes are absent so that one has the simple factorization formula (1.18). In Sec. VII we describe a model where the factorization relation (1.18) can be checked explicitly.

H. Schrödinger equation versus path integral

One can use two methods for the analysis of the models for the baryon wave function: the stationary Schrödinger equation or the path integral approach. In principle, these two methods should give equivalent results. It is well known that the large-N_c limit has many common features with the semiclassical limit. In the path integral approach this leads to a representation for $W(g)$ in terms of classical *time-dependent* trajectories. In the case of the stationary Schrödinger equation one works with the *time-independent* formalism.

In the simplest models (i.e. models with the trivial vacuum), the connection between the two representations can be easily seen. However, in the general case the equivalence of results obtained in the two approaches is less obvious. Therefore in this paper we use both methods. The first part of the paper (Secs. II–VIII) is based on the Schrödinger equation. In Sec. IX we describe the path integral approach and derive a representation for $W(g)$ in terms of trajectories. After that we demonstrate the equivalence of the two representations for $W(g)$ in Sec. X.

I. Large-N_c limit and classical dynamics

An essential part of this paper is devoted to the representation of $W(g)$ in terms of classical trajectories described by an effective Hamiltonian. In fact, the functional $W(g)$ can be interpreted as a classical action depending on the "coordinate" g. This action obeys the classical Hamilton–Jacobi equation. The method of trajectories can be used for the construction of the solution of this Hamilton–Jacobi equation.

The connection between the large-N systems and classical dynamics was extensively discussed in literature [21] (see also review [22] and references therein). In the path integral approach to the large-N systems, the $1/N$ expansion can be constructed using the saddle point method. In a straightforward approach the corresponding saddle point equations have a nonlocal form. Nevertheless in many cases it is possible to find new degrees of freedom which reduce the saddle point equations to a local form described by a local Hamiltonian dynamics.

In this paper we use similar methods and reduce the problem of the calculation of the functional $W(g)$ to the analysis of trajectories described by an effective Hamiltonian. However, the problem studied in this paper is different from the "traditional" time-dependent problems in large-N systems [21]. Most of the works on the effective classical dynamics of large-N_c systems concentrate on the time dependence of *expectation values* of observables so that the classical trajectories correspond to the time-dependent *diagonal* matrix elements of the corresponding quantum operators. On the contrary, the calculation of the baryon wave function is a problem of *nondiagonal transition matrix elements.* In spite of the difference between the two cases, we will see that the corresponding effective classical dynamical systems have many common features.

J. Large N_c models and traditional many-body physics

In our analysis of the large-N_c models we will meet many equations which are well known in the "traditional" many-body physics (see e.g. Ref. [23]). The machinery of the old many-body physics includes many interesting equations: stationary Hartree–Fock, time-dependent Hartree–Fock equation, random phase approximation (RPA) equations. A common feature of these equations is that their derivation is usually based on *ad hoc* approximations so that in a general case the validity of these approximations is a matter of luck. For example, the standard derivation of the Hartree–Fock equation is based on the variational principle with a special ansatz for the wave function. Nevertheless in some cases one can find a justification for these approximations. In particular, the $1/N$ expansion (N not necessarily being color) puts these equations on solid ground [24, 25]. In the framework of the $1/N$ expansion, the Hartree equation corresponds to the leading order of the $1/N$ expansion whereas the contribution of the Fock term is $1/N$ suppressed. Similarly, in the next-to-leading order of the $1/N$ expansion one arrives at the RPA equation [24, 26] with slight modifications caused by the large-N limit

As was explained above, the functional $W(g)$ cannot be expressed via the solution of the *stationary* Hartree equation. However, in Sec. VI we will see that $W(g)$ can be expressed via solutions of the so-called *time-dependent* Hartree equation (TDHE). Indeed, as was mentioned above, the functional $W(g)$ is just an action for some effective mechanical system. The equations of motion describing these trajectories of this mechanical system are similar to the TDHE. The properties and solutions of the TDHE were extensively studied in the context of the problems of the many-body physics (typically in its Hartree–Fock version [23]). Strictly speaking, the analogy between our equations and TDHE is not complete since our problem the large-N_c wave functions differs from the traditional problems of the many-body physics. Nevertheless many interesting connections can be found. For example, our analysis of the effective large-N_c dynamics makes use of the Hamiltonian structure of the classical equations of motion. In the context of the "traditional" approach to TDHE (with Fock term included), the classical Hamiltonian structure of the TDHE was studied in Ref. [27].

K. Antiquarks in large-N_c models

In the simplest models the baryon consists of N_c quarks without any quark-antiquark pairs. This class of models is studied in the first part of the paper (Secs. III –VII). The second part of this paper (Secs. VIII–X) deals more complicated models where the quark-antiquark pairs are described using the well-known "old-fashioned" Dirac representation for antiparticles. In these models we have a nontrivial *physical* vacuum made of MN_c quarks put into the *bare* vacuum (with some integer M), whereas the baryon is considered as a state containing $(M+1)N_c$ quarks above the bare vacuum. The baryon wave function [defined by Eq. (1.2) as a transition matrix element between the baryon and the *physical* vacuum] still can be used for the construction of the generating functional $\Phi_B(g)$ (1.5). We show that in this class of models we also have the exponential behavior (1.6) of $\Phi_B(g)$ and develop methods for the calculation of the functional $W(g)$.

L. From quantum mechanics to quantum field theory

The general structure of the $1/N_c$ expansion for the functional $\Phi_B(g)$ is the same in quantum mechanics and in quantum field theory. For simplicity we use the *discrete* quantum mechanical notation in this paper. For example, instead of the continuous variables x_k of Eq. (1.5) we write discrete indices i_k:

$$\Phi_B(g) = \sum_{i_1 i_2 \ldots i_{N_c}} \Psi^B_{i_1 i_2 \ldots i_{N_c}} g_{i_1} g_{i_2} \cdots g_{i_k} \,. \tag{1.19}$$

This compact discrete notation allows us to simplify equations. By inertia we will often call $\Phi_B(g)$ generating *functional,* although the word *function* could be more appropriate.

One can ask whether our compact discrete notation is only a matter of language or some serious problems will come in the case of field theoretical models. The main problem is that our quantum-mechanical models are based on the four-fermion interaction. Their field theory analog of these models is represented by model of the Nambu–Jona-Lasinio (NJL) type. Formally our quantum-mechanical equations can be generalized for case of the field theory. However, there is a problem with the nonrenormalizability of the four-fermion interaction. From the physical point of view, models of the NJL type can be considered only as low-energy effective models which should be simply cut at high momenta. But problem is that our treatment is based on the Hamiltonian (or path integral) formalism which assumes the locality in time. Therefore applications of our formalism to NJL and similar models would require regularizations preserving the locality in time. Unfortunately most of regularization local in time conflict with the Lorentz invariance.

It should be stressed that the methods developed in this paper can be applied only to the large-N_c models of QCD but not for QCD itself. Nevertheless some special problems of large-N_c QCD these methods still can be used::

1) the problem of heavy-quark baryons [4],

2) the problem of the diagonalization of the matrix of anomalous dimensions for the leading-twist baryon operators [16].

M. Structure of the paper

As was already mentioned, the paper consists of two parts. The first part (Secs. III–VII) is devoted to models with the trivial vacuum (i.e. without antiquarks) whereas in Secs. VIII–X more powerful methods are developed for models imitating antiquarks in terms of the old Dirac picture. The general class of models is discussed in Sec. II. In Sec. III we describe the simplest models of the baryon wave function where the baryon appears as a bound state of N_c quarks (without quark-antiquark pairs). In Sec. IV we derive a nonlinear differential equation for the functional $W(g)$ and show that this equation agrees with the standard Hartree equation for large-N_c systems. In Sec. V we turn to the analysis of the preexponential functional $A(g)$ (1.7). In Sec. VI we show how the problem of the calculation of $W(g)$ can be reduced to the analysis of trajectories in some effective classical dynamics. In Sec. VII we study a simple model where the functional $W(g)$ can be computed analytically. In Sec. VIII we turn to systems with the nontrivial vacuum, using the Schrödinger equation. In Sec. IX we describe the path integral approach to the large-N_c models of the baryon wave function. In Sec. X we check the agreement of the results based on the Schrödinger equation and on the path integral approach.

II. MODELS

A. Choice of models

As was explained in the introduction, our aim is to check conjectures about the structure of the $1/N_c$ expansion for the baryon wave function which were put forward in Refs. [16], [17]. We want to test these conjectures in simple large-N_c models. Our choice of these models is determined by the balance between the intention to preserve the main features of the $1/N_c$ expansion in QCD and the possibility to solve the model. As was explained in the introduction, the aspects of the phenomenological relevance will play a secondary role (if any) in our choice of models.

The minimal constraints on the models include two conditions: the model must have

1) quarks degrees of freedom,

2) $SU(N_c)$ symmetry.

The quark degrees of freedom are understood here in a rather weak sense. Neither full-fledged quantum field theory nor complete quantum mechanics is needed for our aims. For example, we can work with models ignoring the space motion of quarks where the dynamics of quarks is described only by color and some other discrete quantum numbers (e.g. spin and flavor). For the construction of the $1/N_c$ expansion we need the global $SU(N_c)$ symmetry but there is no need in the local gauge invariance.

Now let us turn to the dynamics of the models. Most of our work will be done in the Hamiltonian approach based on the analysis of the stationary Schrödinger equation. In principle, we can include both quarks and antiquarks in our models using the old Dirac picture. In this case the nontrivial physical vacuum will contain the "Dirac" sea of quarks above the vacuum and we must solve the Schrödinger equation both for the vacuum $|0\rangle$ and for the baryon

$|B\rangle$

$$H|0\rangle = \mathcal{E}_{\rm vac}|0\rangle \ , \tag{2.1}$$
$$H|B\rangle = \mathcal{E}_B|B\rangle \, . \tag{2.2}$$

The dynamics of our models will be formulated in terms of the fermionic annihilation and creation operators a_{nc}, a^+_{nc}

$$\{a_{nc}, a^+_{n'c'}\} = \delta_{nn'}\delta_{cc'} \, , \tag{2.3}$$

where

$$c = 1, 2, \ldots, N_c \tag{2.4}$$

is the color index and $n = 1, \ldots, K$ is some index which may have the meaning of spin, isospin, etc.

We will be interested in the baryon wave function given by the transition matrix element

$$\langle 0| \prod_{c=1}^{N_c} a_{m_c c}|B\rangle \, . \tag{2.5}$$

where a_{mc} are quark annihilation operators. We can define the generating functional for this wave function by analogy with Eq. (1.5)

$$\Phi_B(g) = \langle 0| \prod_{c=1}^{N_c} \left(\sum_m g_m a_{mc} \right) |B\rangle \, . \tag{2.6}$$

At large N_c we expect the exponential behavior (1.6)

$$\Phi_B(g) \overset{N_c \to \infty}{\sim} \exp\left[N_c W(g)\right] \, . \tag{2.7}$$

B. Hamiltonian

We will work with models described by the Hamiltonian

$$H = \frac{1}{2N_c} \sum_{n_1 n_2 n_3 n_4} V_{n_1 n_2 n_3 n_4} \left(\sum_{c'=1}^{N_c} a^+_{n_1 c'} a_{n_2 c'} \right) \left(\sum_{c=1}^{N_c} a^+_{n_3 c} a_{n_4 c} \right) + \sum_{n_1 n_2} L_{n_1 n_2} \left(\sum_{c=1}^{N_c} a^+_{n_1 c} a_{n_2 c} \right) \tag{2.8}$$

assuming that

$$V_{n_1 n_2 n_3 n_4} = V_{n_3 n_4 n_1 n_2} \, . \tag{2.9}$$

We we will be interested in color singlet states. The color singlet states can contain MN_c "quarks" where M is an integer number. For any fixed M we can study the problem of the lowest color-singlet state containing MN_c fermions.

In our models of the baryon function we associate notation M with the vacuum whereas the baryon will correspond to $M + 1$ so that one has a nonzero matrix element (2.5) which will be interpreted as the baryon wave function.

C. Hartree equation

Below we will study the limit of large N_c at fixed values of the parameters $V_{n_1 n_2 n_3 n_4}$ and $L_{n_1 n_2}$ in the Hamiltonian (2.8). It is well known (see e.g. [24, 26]) that in the leading order of the $1/N_c$ expansion the energy of the ground state can be found by solving the Hartree equation

$$\sum_{n_2} h_{n_1 n_2} \phi^s_{n_2} = \varepsilon_a \phi^s_{n_1} \, , \tag{2.10}$$

where the single particle Hamiltonian $h_{n_1n_2}$ is given by

$$h_{n_1n_2} = \sum_{s=1}^{M} \sum_{n_3n_4} V_{n_1n_2n_3n_4} \phi^{s*}_{n_3} \phi^{s}_{n_4} + L_{n_1n_2} \tag{2.11}$$

and the single particle eigenstates are normalized by the condition

$$\sum_{n} \phi^{r}_{n} \phi^{s*}_{n} = \delta^{rs} \,. \tag{2.12}$$

In the leading order of the $1/N_c$ expansion the energy of the corresponding state is

$$\mathcal{E} = N_c E_0 + O(N_c^0) \,, \tag{2.13}$$

where E_0 is given by

$$E_0 = \frac{1}{2} \sum_{r,s=1}^{M} \sum_{n_1n_2n_3n_4} V_{n_1n_2n_3n_4} \phi^{r*}_{n_1} \phi^{r}_{n_2} \phi^{s*}_{n_3} \phi^{s}_{n_4} + \sum_{s=1}^{M} \sum_{n_1n_2} L_{n_1n_2} \phi^{s*}_{n_1} \phi^{s}_{n_2} \,. \tag{2.14}$$

In the case of Hamiltonians with $L_{n_1n_2} = 0$ the expression for E_0 can be simplified using Eq. (2.10):

$$E_0 = \frac{1}{2} \sum_{s=1}^{M} \varepsilon_s \quad (\text{if } L_{mn} = 0) \,. \tag{2.15}$$

Below we will see that the knowledge of the solutions of the Hartree equations for the vacuum $|0\rangle$ and for the baryon $|B\rangle$ is not sufficient for the calculation of the generating functional $W(g)$ appearing in Eq. (2.7). One has to derive a special equation for $W(g)$.

D. Time-dependent Hartree equation

The time-dependent Hartree equation (TDHE) appears in various problems of the traditional many-body physics (where its Hartree–Fock modification is usually discussed, see e.g. [23]). In the case of the Hamiltonian (2.8) TDHE has the form

$$i \frac{d}{dt} \phi^{s}_{n_1}(t) = \sum_{n_2} h_{n_1n_2}(t) \phi^{s}_{n_2}(t) \,, \tag{2.16}$$

where

$$h_{n_1n_2}(t) = \sum_{r=1}^{M} \sum_{n_3n_4} V_{n_1n_2n_3n_4} \phi^{r*}_{n_3}(t) \phi^{r}_{n_4}(t) + L_{n_1n_2} \,. \tag{2.17}$$

Although the context of the traditional problems of the many-body physics where TDHE appears differs from our problem of the calculation of the functional $W(g)$, we will see in Sec. VI B that the functional $W(g)$ allows for an interesting representation in terms of the solutions of the TDHE (2.16), (2.17). The possibility of the Hamiltonian interpretation of the Hartree–Fock equations was discussed in Ref. [27]. This Hamiltonian interpretation plays an important role in our analysis.

III. MODELS WITH THE TRIVIAL VACUUM

A. Simplest models

As was explained in Sec. II A, we want to study the baryon wave function in models described by the Hamiltonian (2.8) using the states containing N_cM and $N_c(M+1)$ fermions for the vacuum and baryon, respectively.

We want to start from the simplest case $M = 0$ (the generalization to arbitrary M will be considered in Secs. VIII–X). In these $M = 0$ models the physical vacuum coincides with the bare one, and the baryon consists of N_c

quarks only. We will refer to the $M = 0$ models as *models with the trivial vacuum.* The main advantage of these models is that the solution $\Psi_{\rm bar}$ of the Schrödinger equation (2.2) directly gives us the baryon wave function (2.5).

Our first aim is to define the generating functional $\Phi_B(g)$ for the baryons in these models and to rewrite the stationary Schrödinger equation in terms of $\Phi_B(g)$. This can be easily done using the well known holomorphic representation (also known as boson representation in the context of the many-body physics [23]).

In our simple models "baryons" are described by states containing N_c quarks:

$$\begin{aligned}|\psi^B\rangle &= \sum_{n_1 n_2 \dots n_{N_c}} \psi^B_{n_1 \dots n_{N_c}} a^+_{n_{N_c} N_c} \dots a^+_{n_1 1} |0\rangle \\ &= \frac{1}{N_c!} \sum_{n_1 n_2 \dots n_{N_c}} \sum_{c_1 c_2 \dots c_{N_c}} \varepsilon_{c_1 \dots c_{N_c}} \psi^B_{n_1 \dots n_{N_c}} a^+_{n_{N_c} c_{N_c}} \dots a^+_{n_1 c_1} |0\rangle \,. \end{aligned} \tag{3.1}$$

Here $\psi_{n_1 \dots n_{N_c}}$ are coefficients. Due to the antisymmetry of fermionic operators coefficients $\psi_{n_1 \dots n_{N_c}}$ are completely symmetric in all indices $n_1 \dots n_{N_c}$.

B. Generating function for baryon wave function

In our simple model the baryon wave function (2.5) coincides with the coefficients $\psi^B_{n_1 \dots n_{N_c}}$ of the decomposition (3.1):

$$\psi^B_{n_1 \dots n_{N_c}} = \langle 0 | a^+_{n_1 1} \dots a^+_{n_{N_c} c_{N_c}} | \psi^B \rangle \,. \tag{3.2}$$

Now we can construct the model analog of the generating functional (1.5) describing the baryon wave function:

$$\Phi_B(g) = \sum_{n_1 n_2 \dots n_{N_c}} \psi^B_{n_1 \dots n_{N_c}} g_{n_1} \dots g_{n_{N_c}} = \sum_{n_1 n_2 \dots n_{N_c}} \langle 0 | a^+_{n_1 1} \dots a^+_{n_{N_c} c_{N_c}} | \psi^B \rangle g_{n_1} \dots g_{n_{N_c}} \,. \tag{3.3}$$

Note that function $\Phi_B(g)$ (3.3) is a homogeneous polynomial of degree N_c:

$$\Phi_B(\lambda g) = \lambda^{N_c} \Phi_B(g) \,. \tag{3.4}$$

The correspondence between the states $|\psi^B\rangle$ and functions $\Phi_B(g)$ is useful not only for the description of the eigenstates of the Hamiltonian (2.8) but for any color singlet states containing N_c quarks. Let us consider operator

$$T_{mn} = \sum_{c=1}^{N_c} a^+_{mc} a_{nc} \tag{3.5}$$

on the state $|\psi\rangle$. In terms of the $\Phi_B(g)$ representation for states $|\psi^B\rangle$

$$|\psi^B\rangle \to \Phi_B(g) \tag{3.6}$$

operator T_{mn} takes the form

$$T_{mn} = \sum_{c=1}^{N_c} a^+_{mc} a_{nc} \quad \to \quad g_m \frac{\partial}{\partial g_n} \,. \tag{3.7}$$

C. Schrödinger equation for $\Phi_B(g)$

Using the g representation (3.7) for the operators T_{mn}, we can rewrite the Hamiltonian (2.8) in the form

$$H = \frac{1}{2N_c} \sum_{n_1 n_2 n_3 n_4} V_{n_1 n_2 n_3 n_4} \left(g_{n_1} \frac{\partial}{\partial g_{n_2}} \right) \left(g_{n_3} \frac{\partial}{\partial g_{n_4}} \right) + \sum_{n_1 n_2} L_{n_1 n_2} \left(g_{n_1} \frac{\partial}{\partial g_{n_2}} \right) . \tag{3.8}$$

Then the Schrödinger equation

$$H |\psi^B\rangle = \mathcal{E}_B |\psi^B\rangle \tag{3.9}$$

becomes

$$\frac{1}{2N_c}\sum_{n_1n_2n_3n_4} V_{n_1n_2n_3n_4}\left(g_{n_1}\frac{\partial}{\partial g_{n_2}}\right)\left(g_{n_3}\frac{\partial}{\partial g_{n_4}}\right)\Phi_B(g) + \sum_{n_1n_2} L_{n_1n_2}\left(g_{n_1}\frac{\partial}{\partial g_{n_2}}\right) = \mathcal{E}_B\Phi_B(g)\,. \tag{3.10}$$

IV. LARGE-N_c LIMIT

A. Two approaches to the $1/N_c$ expansion

In this section we want to study the large-N_c limit in the simplest models of baryons introduced in Sec. III. As was mentioned in the introduction, one can use two approaches to the large-N_c limit in fermion systems:

1) approach based on the single-particle *density matrix* obeying the Hartree equation,

2) approach based on the baryon *wave function* described in terms of the generating functional $\Phi_B(g)$.

In principle these two methods are equivalent. For example, they lead to the same $1/N_c$ expansion for the spectrum of states. The second method based on the functional $\Phi_B(g)$ provides more information. However, one has to pay some price for the additional information: the large-N_c equations derived for the functional $\Phi_B(g)$ are more complicated than the Hartree equation for the density matrix.

Below we consider both approaches and demonstrate their equivalence.

B. Large N_c limit for functionals $\Phi_B(g)$

We want to study the limit of $\Phi_B(g)$ at large N_c. According to Eq. (3.4) we have

$$W(\lambda g) = W(g) + \ln\lambda\,, \tag{4.1}$$

$$A_B(\lambda g) = A_B(g)\,, \tag{4.2}$$

so that

$$\sum_n g_n\frac{\partial}{\partial g_n}W(g) = 1\,, \tag{4.3}$$

$$\sum_n g_n\frac{\partial}{\partial g_n}A_B(g) = 0\,. \tag{4.4}$$

Let us insert Eq. (1.7) into the Schrödinger equation (3.10). Taking into account the $1/N_c$ expansion for the energy

$$\mathcal{E}_B = N_cE_0 + \Delta E_B + O(N_c^{-1})\,, \tag{4.5}$$

we obtain in the leading order

$$\frac{1}{2}\sum_{n_1n_2n_3n_4} V_{n_1n_2n_3n_4}\left[g_{n_1}\frac{\partial W(g)}{\partial g_{n_2}}\right]\left[g_{n_3}\frac{\partial W(g)}{\partial g_{n_4}}\right] + \sum_{n_1n_2} L_{n_1n_2}\left[g_{n_1}\frac{\partial W(g)}{\partial g_{n_2}}\right] = E_0\,. \tag{4.6}$$

In the next order of the $1/N_c$ expansion we find

$$\sum_{n_1n_2n_3n_4} V_{n_1n_2n_3n_4}\left[g_{n_1}\frac{\partial W(g)}{\partial g_{n_2}}\right]\left[g_{n_3}\frac{\partial \ln A_B(g)}{\partial g_{n_4}}\right] + \sum_{n_1n_2} L_{n_1n_2}\left[g_{n_1}\frac{\partial \ln A_B(g)}{\partial g_{n_2}}\right] + \frac{1}{2}\sum_{n_1n_2n_3n_4} V_{n_1n_2n_3n_4}\left[g_{n_1}\frac{\partial}{\partial g_{n_2}}\left(g_{n_3}\frac{\partial W(g)}{\partial g_{n_4}}\right)\right] = \Delta E_B\,. \tag{4.7}$$

C. Hartree equation

As was explained in Sec. II C, the energy of the lowest baryon state can be described in the leading order of the $1/N_c$ expansion by the Hartree equations (2.10), (2.11). Our case corresponds to $M = 1$ in these equations:

$$\sum_{n_2} \left(\sum_{n_3 n_4} V_{n_1 n_2 n_3 n_4} \phi^*_{n_3} \phi_{n_4} + L_{n_1 n_2} \right) \phi_{n_2} = \varepsilon_0 \phi_{n_1} \,, \tag{4.8}$$

$$\sum_n \phi^*_n \phi_n = 1 \,, \tag{4.9}$$

where ϕ_n is the single-particle wave function of the level occupied with N_c quarks.

The leading order contribution $N_c E_0$ to the energy $\mathcal{E}_B$ (4.5) is given by Eq. (2.14)

$$E_0 = \frac{1}{2} \sum_{n_1 n_2 n_3 n_4} V_{n_1 n_2 n_3 n_4} \phi^*_{n_1} \phi_{n_2} \phi^*_{n_3} \phi_{n_4} + \sum_{n_1 n_2} L_{n_1 n_2} \phi^*_{n_1} \phi_{n_2} \,. \tag{4.10}$$

D. Agreement between the Hartree equation and the equation for $W(\phi)$

Now we can compare equation (4.6) for $W(g)$ with the expression (4.10) for the Hartree energy E_0. We see that the two expressions agree if we take

$$\phi_n = \frac{\partial W(g)}{\partial g_n} \,, \quad \phi^*_n = g_n \,. \tag{4.11}$$

In other words, if we know function $W(g)$, then the Hartree solution ϕ_n can be found by solving the equation

$$g^*_n = \frac{\partial W(g)}{\partial g_n} \quad \Longrightarrow \phi_n = g^*_n \,. \tag{4.12}$$

Note that the normalization condition (4.9) is automatically satisfied

$$\sum_n \phi^*_n \phi_n = \sum_n g_n \frac{\partial W(g)}{\partial g_n} = 1 \tag{4.13}$$

due to the property (4.3) of the function $W(g)$.

E. Normalization integral

Let us consider two states (3.1)

$$|\psi^{(k)}\rangle = \sum_{n_1 n_2 \ldots n_{N_c}} \psi^{(k)}_{n_1 \ldots n_{N_c}} a^+_{n_{N_c} N_c} \ldots a^+_{n_1 1} |0\rangle \quad (k = 1, 2) \tag{4.14}$$

described by functions (3.3)

$$\Phi^{(k)}(g) = \sum_{n_1 n_2 \ldots n_{N_c}} \psi^{(k)}_{n_1 \ldots n_{N_c}} g_{n_1} \ldots g_{n_{N_c}} \,. \tag{4.15}$$

Then

$$\begin{aligned}
\langle \psi^{(1)} | \psi^{(2)} \rangle &= \sum_{i_1 i_2 \ldots i_{N_c}} F^{(1)*}_{i_1 i_2 \ldots i_{N_c}} F^{(2)}_{i_1 i_2 \ldots i_{N_c}} = \left(\frac{1}{N_c!} \right)^2 \left(\frac{\partial}{\partial g_i} \frac{\partial}{\partial g^*_i} \right)^{N_c} \left[\Phi^{(1)}(g) \right]^* \left[\Phi^{(2)}(g) \right] \\
&= \frac{N_c^{N_c}}{N_c!} \exp \left(\frac{1}{N_c} \frac{\partial}{\partial g_i} \frac{\partial}{\partial g^*_i} \right) \left[\Phi^{(1)}(g) \right]^* \left[\Phi^{(2)}(g) \right] \Big|_{g = g^* = 0} \\
&= \frac{N_c^{N_c}}{N_c!} \frac{\int dg dg^* \exp \left(-N_c \sum_i g_i g^*_i \right) \left[\Phi^{(1)}(g) \right]^* \left[\Phi^{(2)}(g) \right]}{\int dg dg^* \exp \left(-N_c \sum_i g_i g^*_i \right)} \,.
\end{aligned} \tag{4.16}$$

At large N_c we can use the asymptotic expression (1.7) for $\Phi^{(k)}(g)$. As a result, we find

$$\langle\psi^{(1)}|\psi^{(2)}\rangle \to \frac{e^{N_c} N_c^{\nu_1+\nu_2}}{\sqrt{2\pi N_c}} \frac{\int dg dg^* \exp\left\{N_c\left\{-g_i g_i^* + [W(g)]^* + W(g)\right\}\right\} \left[A^{(1)}(g)\right]^* A^{(2)}(g)}{\int dg dg^* \exp\left(-N_c \sum_i g_i g_i^*\right)} . \tag{4.17}$$

This integral can be computed using the saddle point method. The saddle point equation is nothing else Eq. (4.12) with the solution $g_n = \phi_n^*$ given by the Hartree equation (4.8). Thus we find with the exponential accuracy

$$\langle\psi^{(1)}|\psi^{(2)}\rangle \sim \exp\left\{-N_c\left\{[W(\phi^*)]^* + W(\phi^*)\right\}\right\} . \tag{4.18}$$

We cannot use the saddle point method beyond this exponential accuracy because of the noncommutativity of the limits

$$g_n \to \phi_n^* , \quad N_c \to \infty \tag{4.19}$$

which is discussed in Sec. V B.

Taking a normalizable state

$$\psi^{(1)} = \psi^{(2)} \equiv \psi , \tag{4.20}$$

$$\langle\psi|\psi\rangle = 1 , \tag{4.21}$$

we conclude from (4.18) that

$$\mathrm{Re}\, W(\phi^*) = 0 . \tag{4.22}$$

V. LARGE N_c LIMIT AND RPA EQUATIONS

A. Beyond the leading order of the $1/N_c$ expansion

In the previous section we have established a connection between the traditional description of the large-N_c systems in terms of the Hartree equation (4.8) and our equation (4.6) for the functional $W(g)$. As is well known, the Hartree equation determines only the leading $O(N_c)$ part of the energy of the states. If one is interested in the $O(N_c^0)$ corrections, then one has to solve the random phase approximation (RPA) equations [24, 26].

On the other hand, in the problem of the baryon wave function the analysis of the next-leading-order of the $1/N_c$ expansion is based on Eq. (4.7) for the functional $A_B(g)$. Therefore one can expect that there must be some connection between the RPA equations and Eq. (4.7). From our experience with the leading order equations [the Hartree equation (4.8) and Eq. (4.6) for $W(g)$] we know that this connection may be rather nontrivial. Indeed, we will see that the RPA equation really appears in the analysis of the $1/N_c$ expansion for the basic functional $\Phi_B(g)$ but in a modified version of the large-N_c limit when the argument g of the functional $\Phi_B(g)$ changes with N_c in a special way. This new version of the large-N_c limit is discussed in Sec. V B. In Sec. V D we turn to the derivation of the RPA equations corresponding to the modified large-N_c limit.

B. Two versions of the large-N_c limit

Eq. (1.7) describes the asymptotic behavior of $\Phi(g)$ in the large-N_c limit when g is fixed.

$$g = \text{const}, \; N_c \to \infty . \tag{5.1}$$

But we can also study another limit

$$N_c \to \infty , \quad g_n = \phi_n + \frac{1}{\sqrt{N_c}} \Delta_n , \quad \Delta_n = \text{const} . \tag{5.2}$$

when g_n approaches the Hartree solution ϕ_n in the limit $N_c \to \infty$. In addition let us impose the condition on Δ_n

$$\sum_n \phi_n^* \Delta_n = 0 . \tag{5.3}$$

In the limit (5.2) we have the asymptotic behavior

$$\Phi_B\left(\phi_n + \frac{1}{\sqrt{N_c}}\Delta_n\right) \overset{N_c\to\infty}{\longrightarrow} N_c^{\nu_B} C_B(\Delta_n)\exp\left[N_c W(\phi)\right] \tag{5.4}$$

different from Eq. (1.7). Taking $\Delta_n = 0$ and comparing the asymptotic expression (5.4) with Eq. (1.7), we obtain

$$C_B(0) = A_B(\phi)\,. \tag{5.5}$$

Replacing $\Delta_n \to \lambda^{-1}\Delta_n$ in Eq. (5.4) and using relation (3.4), we find

$$\begin{aligned}\Phi_B\left(\lambda\phi_n + \frac{1}{\sqrt{N_c}}\Delta_n\right) &= \lambda^{N_c}\Phi_B\left(\phi_n + \frac{1}{\sqrt{N_c}}\lambda^{-1}\Delta_n\right)\\ &\overset{N_c\to\infty}{\longrightarrow} N_c^{\nu_B} C_B(\lambda^{-1}\Delta_n)\exp\left\{N_c\left[W(\phi)+\ln\lambda\right]\right\}\,.\end{aligned} \tag{5.6}$$

C. Special choice of basis

It is convenient to choose the basis diagonalizing the Hartree Hamiltonian such that

$$\phi_n^s = \delta_n^s\,. \tag{5.7}$$

In models with one occupied level we will label this level with the index $s = 0$ and use the short notation

$$\phi_n \equiv \phi_n^0 = \delta_n^0\,. \tag{5.8}$$

Then the Hartree Hamiltonian (2.11) becomes

$$h_{mn} = V_{mn00} + L_{mn} \tag{5.9}$$

and the Hartree equation (4.8) takes the form

$$V_{mn00} + L_{mn} = \varepsilon_m\delta_{mn}\,. \tag{5.10}$$

In particular,

$$\varepsilon_0 = V_{0000} + L_{00}\,. \tag{5.11}$$

The Hartree energy is

$$E_0 = \frac{1}{2}V_{0000} + L_{00}\,. \tag{5.12}$$

We introduce a special notation for the components g_n with $n > 0$:

$$\tilde{g}_n \overset{n>0}{=} g_n\,, \quad \tilde{g}_0 = 0 \tag{5.13}$$

so that

$$g_n = g_0\phi_n + \tilde{g}_n\,, \tag{5.14}$$

$$\Phi_B\left(g\right) = \Phi_B\left(g_0, \tilde{g}\right)\,. \tag{5.15}$$

Now we find from Eq. (5.6)

$$\left[\Phi_B\left(g\right)\right]_{\tilde{g}_n=\Delta_n/\sqrt{N_c}} = \Phi_B\left(g_0\phi_n + \frac{\Delta_n}{\sqrt{N_c}}\right) \overset{N_c\to\infty,\Delta_n=\text{const}}{\longrightarrow} N_c^{\nu_B} C_B(g_0^{-1}\Delta_n)\exp\left\{N_c\left[W(\phi)+\ln g_0\right]\right\}\,. \tag{5.16}$$

Therefore

$$g_0\frac{\partial}{\partial g_0}\left[\Phi_B\left(g\right)\right]_{\tilde{g}_n=\Delta_n/\sqrt{N_c}} \overset{N_c\to\infty,\Delta_n=\text{const}}{\longrightarrow} g_0\frac{\partial}{\partial g_0}N_c^{\nu_B}C_B(g_0^{-1}\Delta_n)\exp\left\{N_c\left[W(\phi)+\ln g_0\right]\right\}$$
$$=\left\{N_c^{\nu_B}\exp\left\{N_c\left[W(\phi)+\ln g_0\right]\right\}\right\}N_c\left(1-\frac{1}{N_c}\sum_{n>0}\tilde{\Delta}_n\frac{\partial}{\partial\tilde{\Delta}_n}\right)C_B(\tilde{\Delta}_n)\Big|_{\tilde{\Delta}_n=g_0^{-1}\Delta_n}. \tag{5.17}$$

On the LHS the differential operator $g_0\partial/\partial g_0$ acts on $\Phi_B\left(g\right)$ whereas on the RHS we have the corresponding operator acting on $C_B(\tilde{\Delta}_n)$. Equation (5.17) means that the following $\tilde{\Delta}$ representation is valid for the operator $g_0\partial/\partial g_0$:

$$g_0\frac{\partial}{\partial g_0}\to N_c\left(1-\frac{1}{N_c}\sum_{n>0}\tilde{\Delta}_n\frac{\partial}{\partial\tilde{\Delta}_n}\right). \tag{5.18}$$

Similarly we find

$$\tilde{g}_m\frac{\partial}{\partial g_0}\to\sqrt{N_c}\tilde{\Delta}_m\left(1-\frac{1}{N_c}\sum_{n>0}\tilde{\Delta}_n\frac{\partial}{\partial\tilde{\Delta}_n}\right), \tag{5.19}$$

$$g_0\frac{\partial}{\partial\tilde{g}_m}\to\sqrt{N_c}\frac{\partial}{\partial\tilde{\Delta}_m}, \tag{5.20}$$

$$g_n\frac{\partial}{\partial\tilde{g}_m}\to\tilde{\Delta}_n\frac{\partial}{\partial\tilde{\Delta}_m}. \tag{5.21}$$

D. RPA equations from the large-N_c limit

Now we can apply the results (5.18)–(5.21) to the Schrödinger equation (3.10). The terms of order N_c generated by $g_0\partial/\partial g_0$ give the Hartree energy (5.12). The terms of order $\sqrt{N_c}$ cancel because of the equation (5.10). The terms of order N_c^0 lead to the equation

$$\sum_{m,n>0}\left[V_{00nm}\left(\tilde{\Delta}_n\frac{\partial}{\partial\tilde{\Delta}_m}\right)+\frac{1}{2}V_{m00n}\left(\tilde{\Delta}_m\frac{\partial}{\partial\tilde{\Delta}_n}+\frac{\partial}{\partial\tilde{\Delta}_n}\tilde{\Delta}_m\right)+\frac{1}{2}V_{0m0n}\frac{\partial}{\partial\tilde{\Delta}_m}\frac{\partial}{\partial\tilde{\Delta}_m}\right.$$
$$\left.+\frac{1}{2}V_{m0n0}\tilde{\Delta}_m\tilde{\Delta}_n+L_{mn}\tilde{\Delta}_m\frac{\partial}{\partial\tilde{\Delta}_n}-\left(V_{0000}+L_{00}\right)\delta_{mn}\tilde{\Delta}_m\frac{\partial}{\partial\tilde{\Delta}_n}\right]C_B(\tilde{\Delta})$$
$$=\Delta E_BC_B(\tilde{\Delta}). \tag{5.22}$$

This equation can be simplified using Eq. (5.11):

$$\sum_{m,n>0}\left[V_{00nm}\left(\tilde{\Delta}_n\frac{\partial}{\partial\tilde{\Delta}_m}\right)+\frac{1}{2}V_{m00n}\left(\tilde{\Delta}_m\frac{\partial}{\partial\tilde{\Delta}_n}+\frac{\partial}{\partial\tilde{\Delta}_n}\tilde{\Delta}_m\right)+\frac{1}{2}V_{0m0n}\frac{\partial}{\partial\tilde{\Delta}_m}\frac{\partial}{\partial\tilde{\Delta}_m}\right.$$
$$\left.+\frac{1}{2}V_{m0n0}\tilde{\Delta}_m\tilde{\Delta}_n+L_{mn}\tilde{\Delta}_m\frac{\partial}{\partial\tilde{\Delta}_n}-\varepsilon_0\delta_{mn}\tilde{\Delta}_m\frac{\partial}{\partial\tilde{\Delta}_n}\right]C_B(\tilde{\Delta})=\Delta E_BC_B(\tilde{\Delta}). \tag{5.23}$$

Let us introduce operators

$$b_m=\frac{\partial}{\partial\tilde{\Delta}_m}, \tag{5.24}$$

$$b_m^+=\tilde{\Delta}_m, \tag{5.25}$$

$$[b_m,b_n^+]=\delta_{mn}. \tag{5.26}$$

Then

$$\sum_{m,n>0}\left[V_{00nm}b_n^+b_m+\frac{1}{2}V_{m00n}\left(b_m^+b_n+b_nb_m^+\right)+\frac{1}{2}V_{0m0n}b_m^+b_n\right.$$
$$\left.+\frac{1}{2}V_{m0n0}b_m^+b_n^++L_{mn}b_m^+b_n-\varepsilon_0\delta_{mn}b_m^+b_n\right]C_B=\Delta E_BC_B. \tag{5.27}$$

Note that the b_n vacuum $|0\rangle$

$$b_m|0\rangle = 0 \tag{5.28}$$

corresponds in the $\tilde{\Delta}$ representation (5.24), (5.25) to the wave function

$$\Psi(\tilde{\Delta}) \equiv 1\,. \tag{5.29}$$

Thus we have the Hamiltonian

$$\begin{aligned} H_{\rm RPA} \equiv & \sum_{m,n>0} \Big[V_{00nm} b_n^+ b_m + \frac{1}{2} V_{m00n} \left(b_m^+ b_n + b_n b_m^+ \right) + \frac{1}{2} V_{0m0n} b_m b_n \\ & + \frac{1}{2} V_{m0n0} b_m^+ b_n^+ + L_{mn} b_m^+ b_n - \varepsilon_0 \delta_{mn} b_m^+ b_n \Big] \\ = & \frac{1}{2} \sum_{m,n>0} \begin{pmatrix} b_m \\ b_m^+ \end{pmatrix}^T \begin{pmatrix} V_{0m0n} & V_{n00m} + V_{00nm} + L_{nm} - \varepsilon_0 \delta_{mn} \\ V_{m00n} + V_{00mn} + L_{mn} - \varepsilon_0 \delta_{mn} & V_{m0n0} \end{pmatrix} \begin{pmatrix} b_n \\ b_n^+ \end{pmatrix} \\ & - \frac{1}{2} \sum_{m>0} \left[(V_{00mm} + L_{mm}) - \varepsilon_0 \right] . \end{aligned} \tag{5.30}$$

According to Eqs. (2.9) and (5.10) we have

$$V_{00nm} + L_{nm} = \varepsilon_m \delta_{nm}\,, \tag{5.31}$$

$$\sum_{m>0} (V_{00mm} + L_{mm}) = \sum_{m>0} \varepsilon_m\,. \tag{5.32}$$

Now we find

$$H_{\rm RPA} = \frac{1}{2} \sum_{m,n>0} \begin{pmatrix} b_m \\ b_m^+ \end{pmatrix}^T \mathcal{R} \begin{pmatrix} b_n \\ b_n^+ \end{pmatrix} - \frac{1}{2} \sum_{m>0} (\varepsilon_m - \varepsilon_0)\,, \tag{5.33}$$

where

$$\mathcal{R} = \begin{pmatrix} V_{0m0n} & V_{n00m} + (\varepsilon_m - \varepsilon_0)\,\delta_{mn} \\ V_{m00n} + (\varepsilon_m - \varepsilon_0)\,\delta_{mn} & V_{m0n0} \end{pmatrix} . \tag{5.34}$$

This is nothing else but the large-N_c version of the RPA Hamiltonian.

E. Diagonalization of the RPA Hamiltonian

We can diagonalize the RPA Hamiltonian (5.33) using the Bogolyubov transformation

$$U b_m U^{-1} = \sum_{n>0} \left(\alpha_{mn} b_n + \beta_{mn} b_n^+ \right)\,, \tag{5.35}$$

$$U b_m^+ U^{-1} = \sum_{n>0} \left(\beta_{mn}^* b_n + \alpha_{mn}^* b_n^+ \right)\,, \tag{5.36}$$

$$U^{-1} b_m U = \sum_{n>0} \left(\alpha_{nm}^* b_n - \beta_{nm} b_n^+ \right), \tag{5.37}$$

$$U^{-1} b_m^+ U = \sum_{n>0} \left(-\beta_{nm}^* b_n + \alpha_{nm} b_n \right)\,. \tag{5.38}$$

Matrices α, β obey the conditions

$$\alpha\alpha^+ - \beta\beta^+ = 1\,, \tag{5.39}$$

$$\alpha\beta^T = \beta\alpha^T\,, \tag{5.40}$$

$$\alpha^+\alpha - \beta^T\beta^* = 1\,, \tag{5.41}$$

$$\alpha^+\beta = \beta^T\alpha^*\,. \tag{5.42}$$

Eqs. (5.39) and (5.40) can be rewritten in the form

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} \alpha^+ & -\beta^T \\ -\beta^+ & \alpha^T \end{pmatrix} = 1\,. \tag{5.43}$$

Introducing the notation

$$\mathcal{S} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \tag{5.44}$$

we can write

$$U \begin{pmatrix} b_m \\ b_m^+ \end{pmatrix} U^{-1} = \left[\mathcal{S} \begin{pmatrix} b \\ b^+ \end{pmatrix} \right]_m . \tag{5.45}$$

Now we can diagonalize the RPA Hamiltonian (5.33) using Eq. (5.45)

$$UH_{\rm RPA}U^{-1} = \frac{1}{2} \sum_{m,n>0} \begin{pmatrix} b_m \\ b_m^+ \end{pmatrix}^T \left(\mathcal{S}^T\mathcal{R}\mathcal{S}\right)_{mn} \begin{pmatrix} b_n \\ b_n^+ \end{pmatrix} - \frac{1}{2} \sum_{m>0} \left(\varepsilon_m - \varepsilon_0\right) . \tag{5.46}$$

Let us choose $\mathcal{S}$ so that

$$\mathcal{R} = \mathcal{S}^T \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \mathcal{S}\,, \tag{5.47}$$

where Ω is a diagonal matrix

$$\Omega = {\rm diag}(\Omega_m)\,. \tag{5.48}$$

Then

$$UH_{\rm RPA}U^{-1} = \frac{1}{2} \sum_{m>0} \left[\Omega_m \left(b_m^+ b_m + b_m b_m^+\right) - \left(\varepsilon_m - \varepsilon_0\right)\right] . \tag{5.49}$$

The spectrum of this Hamiltonian gives the $O(N_c^0)$ contribution ΔE_B to the $1/N_c$ expansion of the total energy (4.5):

$$\Delta E_B = \sum_{m>0} \left[\Omega_m \left(n_m + \frac{1}{2}\right) - \frac{1}{2}\left(\varepsilon_m - \varepsilon_0\right)\right]. \tag{5.50}$$

The corresponding eigenstates of $H_{\rm RPA}$ are according to Eqs. (5.38) and (5.49)

$$U^{-1} \left[\prod_k \left(b_k^+\right)^{n_k}\right] |0\rangle = \left[\prod_k \left(-\sum_{m>0} \beta_{mk}^* b_m + \alpha_{mk} b_m^+\right)^{n_k}\right] U^{-1}|0\rangle\,. \tag{5.51}$$

Here $|0\rangle$ is the b_m vacuum (5.28) and the ground state of $H_{\rm RPA}$ is

$$|0_{\rm RPA}\rangle = U^{-1}|0\rangle\,. \tag{5.52}$$

Using Eqs. (5.37) and (5.28), we find

$$0 = U^{-1}b_m|0\rangle = \left(U^{-1}b_m U\right) U^{-1}|0\rangle = \sum_{n>0} \left(\alpha_{nm}^* b_n - \beta_{nm} b_n^+\right) U^{-1}|0\rangle\,. \tag{5.53}$$

Combining Eqs. (5.52) and (5.53), we obtain

$$\sum_{n>0} \left(\alpha_{nm}^* b_n - \beta_{nm} b_n^+\right) |0_{\rm RPA}\rangle = 0\,. \tag{5.54}$$

F. Calculation of functions $C_B(\tilde{\Delta})$

Functions $C_B(\tilde{\Delta})$ were defined via the asymptotic expression (5.4). Now let us compute these functions. We start from the function $C_0(\tilde{\Delta})$ corresponding to the ground state $|0_{\rm RPA}\rangle$. The ground state $|0_{\rm RPA}\rangle$ obeys Eq. (5.54). In the $\tilde{\Delta}$ representation (5.24), (5.25), Eq. (5.54) takes the form

$$\sum_{n>0}\left(\alpha^*_{nm}\frac{\partial}{\partial\tilde{\Delta}_n}-\beta_{nm}\tilde{\Delta}_n\right)C_0(\tilde{\Delta})=0\,. \tag{5.55}$$

The solution of this equation is

$$C_0(\tilde{\Delta})=\mathcal{N}\exp\left[\frac{1}{2}(\tilde{\Delta}X\tilde{\Delta})\right]\,, \tag{5.56}$$

where

$$(\tilde{\Delta}X\tilde{\Delta})=\sum_{m,n>0}\tilde{\Delta}_m X_{mn}\tilde{\Delta}_n\,, \tag{5.57}$$

$$X_{mn}=\left[\left(\alpha^{*T}\right)^{-1}\beta^T\right]_{mn}=\left[\left(\alpha^{+}\right)^{-1}\beta^T\right]_{mn} \tag{5.58}$$

and $\mathcal{N}$ is a normalization factor.

According to Eq. (5.42) we have

$$\beta\left(\alpha^*\right)^{-1}=\left(\alpha^{+}\right)^{-1}\beta^T\,. \tag{5.59}$$

This means that matrix

$$X_{mn}=\left[\left(\alpha^{+}\right)^{-1}\beta^T\right]_{mn}=\left[\beta\left(\alpha^*\right)^{-1}\right]_{mn} \tag{5.60}$$

is symmetric

$$X_{mn}=X_{nm}\,. \tag{5.61}$$

The excited states (5.51) are described in the $\tilde{\Delta}$ representation (5.24), (5.25) by the wave functions

$$C_B(\tilde{\Delta})=\mathcal{N}\prod_k\left(\sum_{m>0}-\beta^*_{mk}\frac{\partial}{\partial\tilde{\Delta}_m}+\alpha_{mk}\tilde{\Delta}_m\right)^{n_k}\exp\left[\frac{1}{2}(\tilde{\Delta}X\tilde{\Delta})\right]=\mathcal{N}P_B(\tilde{\Delta})\exp\left[\frac{1}{2}(\tilde{\Delta}X\tilde{\Delta})\right]\,. \tag{5.62}$$

Here $P_B(\tilde{\Delta})$ is a polynomial of degree

$$N_B=\sum_k n_k\,. \tag{5.63}$$

At large $\tilde{\Delta}_n$ we have according to Eq. (5.62)

$$P_B(\tilde{\Delta})\stackrel{\tilde{\Delta}\to\infty}{\longrightarrow}\prod_k\left[\left(-\beta^+X\tilde{\Delta}\right)_k\right]^{n_k}\,, \tag{5.64}$$

where

$$\left(-\beta^+X\tilde{\Delta}\right)_k=-\sum_{mn}\beta^*_{mk}X_{mn}\tilde{\Delta}_n\,. \tag{5.65}$$

Using Eqs. (5.60) and (5.41), we find

$$\beta^+X=\beta^+\beta\left(\alpha^*\right)^{-1}=\left(\alpha^T\alpha^*-1\right)\left(\alpha^*\right)^{-1}=\alpha^T-\left(\alpha^*\right)^{-1}\,. \tag{5.66}$$

G. Matching two types of the $1/N_c$ expansion

In Sec. V B we have described two different asymptotic expressions for the large-N_c limit. These two expressions are valid in different regions. However, we can match the two expressions in their common overlap area.

Let us insert Eq. (5.62) into Eq. (5.4)

$$\Phi_B\left(\phi+\frac{\tilde\Delta}{\sqrt{N_c}}\right)=\mathcal{N}P_B(\tilde\Delta)\exp\left[\frac{1}{2}(\tilde\Delta X\tilde\Delta)\right]\exp\left[N_cW(\phi)\right]\quad\left[\tilde\Delta_n=O(N_c^0)\right]. \tag{5.67}$$

This expression must agree with with Eq. (1.7)

$$\Phi_B\left(\phi+\frac{\tilde\Delta}{\sqrt{N_c}}\right)=N_c^{\nu_B}A_B\left(\phi+\frac{\tilde\Delta}{\sqrt{N_c}}\right)\exp\left[N_cW\left(\phi+\frac{\tilde\Delta}{\sqrt{N_c}}\right)\right],\quad\tilde\Delta_n=O(\sqrt{N_c}) \tag{5.68}$$

in the region

$$1\ll\tilde\Delta\ll\sqrt{N_c}\,. \tag{5.69}$$

In this region only the leading term of degree (5.64) survives in the polynomial $P_B(\tilde\Delta)$ so that Eq. (5.67) becomes

$$\Phi_B\left(\phi+\frac{\tilde\Delta}{\sqrt{N_c}}\right)=\mathcal{N}\left\{\prod_k\left[\left(-\beta^+X\tilde\Delta\right)_k\right]^{n_k}\right\}\exp\left[\frac{1}{2}(\tilde\Delta X\tilde\Delta)\right]\exp\left[N_cW(\phi)\right]. \tag{5.70}$$

Now we want to check the general factorization relation (1.18)

$$A_B\left(g\right)=A^{(0)}(g)\prod_k\left[A_k(g)\right]^{n_k}=\tilde A^{(0)}(\tilde g)\prod_k\left[\tilde A_k(\tilde g)\right]^{n_k}. \tag{5.71}$$

If this factorization holds, then Eq. (5.68) takes the form

$$\begin{aligned}\Phi_B\left(\phi+\frac{\tilde\Delta}{\sqrt{N_c}}\right)&=N_c^{\nu_B}\tilde A^{(0)}\left(\frac{\tilde\Delta}{\sqrt{N_c}}\right)\prod_k\left[\tilde A_k\left(\frac{\tilde\Delta}{\sqrt{N_c}}\right)\right]^{n_k}\\&\times\exp\left[N_cW\left(\phi+\frac{\tilde\Delta}{\sqrt{N_c}}\right)\right],\quad\tilde\Delta_n=O(\sqrt{N_c})\end{aligned} \tag{5.72}$$

Due to Eq. (4.11) and condition (5.3),

$$\left(\phi^*\delta\tilde\phi\right)=0\,, \tag{5.73}$$

the variation $W\left(\phi+\delta\tilde\phi\right)$ has no linear term:

$$W\left(\phi+\delta\tilde\phi\right)=W(\phi)+\frac{1}{2}W^{(2)}(\delta\tilde\phi,\delta\tilde\phi)\,. \tag{5.74}$$

Thus

$$\Phi_B\left(\phi+\frac{\tilde\Delta}{\sqrt{N_c}}\right)\approx\tilde A^{(0)}(0)\prod_k\left[A_k\left(\frac{\tilde\Delta}{\sqrt{N_c}}\right)\right]^{n_k}\exp\left[N_cW(\phi)+\frac{1}{2}W^{(2)}(\delta\tilde\Delta,\delta\tilde\Delta)\right]. \tag{5.75}$$

Inserting this into Eq. (5.70), we see that

$$\begin{aligned}&\mathcal{N}\left\{\prod_k\left[\left(-\beta^+X\tilde\Delta\right)_k\right]^{n_k}\right\}\exp\left[\frac{1}{2}(\tilde\Delta X\tilde\Delta)\right]\exp\left[N_cW(\phi)\right]\\&=N_c^{\nu_B}\tilde A^{(0)}(0)\prod_k\left[\tilde A_k\left(\frac{\tilde\Delta}{\sqrt{N_c}}\right)\right]^{n_k}\exp\left[N_cW(\phi)+\frac{1}{2}W^{(2)}(\delta\tilde\Delta,\delta\tilde\Delta)\right].\end{aligned} \tag{5.76}$$

Now we see that

$$\tilde{A}^{(0)}(0) = \mathcal{N} N_c^{-\nu_B + \sum_k n_k/2}\,, \tag{5.77}$$

$$A_k\left(\phi + \delta\tilde{\phi}\right) = \left(-\beta^+ X\,\delta\tilde{\phi}\right)_k\,, \tag{5.78}$$

$$W^{(2)}(\delta\tilde{\phi}, \delta\tilde{\phi}) = \left(\delta\tilde{\phi}\, X\,\delta\tilde{\phi}\right)\,. \tag{5.79}$$

H. Asymptotic behavior in the vicinity of the Hartree solution

Inserting Eq. (5.79) into Eq. (5.74), we find

$$W\left(\phi + \delta\tilde{\phi}\right) = W(\phi) + \frac{1}{2}\left(\delta\tilde{\phi}\, X\,\delta\tilde{\phi}\right) + O\left(\delta\tilde{\phi}^3\right)\,. \tag{5.80}$$

Combining this result with Eq. (4.1), we obtain

$$W\left(\lambda\phi + \delta\tilde{\phi}\right) = W\left(\phi + \lambda^{-1}\delta\tilde{\phi}\right) + \ln\lambda = W(\phi) + \ln\lambda + \frac{1}{2\lambda^2}\left(\delta\tilde{\phi}\, X\,\delta\tilde{\phi}\right) + O\left(\delta\tilde{\phi}^3\right)\,. \tag{5.81}$$

Taking into account Eq. (5.7), we conclude that if

$$|g_m| \ll g_0 \quad (m > 0) \tag{5.82}$$

then

$$W\left(g\right) = W(\phi) + \ln g_0 + \frac{1}{2\left(g_0\right)^2}\sum_{m,n>0} g_m X_{mn} g_n + O\left(\left(\frac{g_{m>0}}{g_0}\right)^3\right)\,. \tag{5.83}$$

In Sec. IV B we derived Eq. (4.6) for the function $W\left(g\right)$. This is a partial differential equation which has many solutions. The asymptotic expansion (5.83) plays the role of the boundary condition for this differential equation. The matrix X_{mn} appearing in this boundary condition is given by Eq. (5.60).

In the same way, using Eq. (4.2), we can rewrite relation (5.78) in the form

$$A_k\left(g\right) = -\frac{1}{g_0}\sum_{m,n>0}\beta^*_{mk} X_{mn} g_n + O\left(\left(\frac{g_{m>0}}{g_0}\right)^2\right)\,. \tag{5.84}$$

This asymptotic behavior of $A_k\left(g\right)$ should be used as the boundary condition for the differential equation (4.7).

VI. CLASSICAL DYNAMICS OF LARGE-N_c SYSTEMS

A. Hamilton–Jacobi equation

According to our previous results, function $W(g)$ can be computed by solving the differential equation (4.6) with the boundary condition (5.83). This representation for $W(g)$ is not quite convenient since one has to deal with the *partial* differential equation (4.6). In this section we show how the problem can be reduced to the analysis of *ordinary* differential equations. This reduction can be done using the well-known Hamilton–Jacobi method.

Eq. (4.6) has the structure of the classical Hamilton–Jacobi equation. In order to make this structure explicit, we change the notation

$$g \to q\,, \tag{6.1}$$

$$W(g) \to S(q)\,. \tag{6.2}$$

Then Eq. (4.6) takes the form

$$H\left(\frac{\partial S(q)}{\partial q}, q\right) = E_0\,, \tag{6.3}$$

where the Hamiltonian is

$$H(p,q) = \frac{1}{2}\sum_{n_1 n_2 n_3 n_4} V_{n_1 n_2 n_3 n_4}\,(q_{n_1} p_{n_2})\,(q_{n_3} p_{n_4}) + \sum_{n_1 n_2} L_{n_1 n_2}\,(q_{n_1} p_{n_2})\,. \tag{6.4}$$

One should keep in mind that g can be complex and $W(g)$ is an analytical function of g (which may have singular and branch points). Therefore our classical mechanics is complex.

As is well known, the solutions of the Hamilton–Jacobi equation can be constructed in terms of the action for the trajectories obeying the Hamiltonian equations

$$\frac{dp_n}{dt} = -\frac{\partial H(p,q)}{\partial q_n}, \quad \frac{dq_n}{dt} = \frac{\partial H(p,q)}{\partial p_n}\,. \tag{6.5}$$

Solution $S(q)$ of the Hamilton–Jacobi equation (6.3) describes the set of trajectories covering the coordinate space. All these trajectories have the same energy E_0 and obey the condition

$$p_n = \frac{\partial S(q)}{\partial q_n}\,. \tag{6.6}$$

B. Static and time-dependent Hartree equations

If we change the notation

$$\phi_n \to p_n\,, \quad \phi_n^* \to q_n \tag{6.7}$$

in the Hartree equation (4.8) and in the normalization condition (4.9) then we obtain

$$\frac{\partial}{\partial p_n} H(p,q) = \varepsilon_0 q_n\,, \tag{6.8}$$

$$\frac{\partial}{\partial q_n} H(p,q) = \varepsilon_0 p_n\,, \tag{6.9}$$

$$\sum_n p_n q_n = 1\,, \tag{6.10}$$

$$p_n = q_n^*\,. \tag{6.11}$$

Let us stress that we consider p_n and q_n as independent complex variables of our phase space. Conditions (6.8)–(6.11) define a point in this space. At this point the variables p_n and q_n are complex conjugate. But generally speaking $p_n \neq q_n^*$.

Strictly speaking, Eqs. (6.8)–(6.11) have more than one solution. For example, these equations are invariant under the phase transformation

$$p_n \to e^{i\alpha} p_n\,, \quad q_n \to e^{-i\alpha} q_n\,. \tag{6.12}$$

In principle, there can be additional symmetries which increase the number of solutions.

If one chooses the basis (5.7), then the solution of the Hartree equations (6.8)–(6.11) is

$$p_n = q_n = \delta_{n0}\,. \tag{6.13}$$

Parameter ε_0 is given by Eq. (5.11).

The change of notation (6.7) in the Hamilton equations (6.5) transforms them into the form

$$\frac{d\phi_m}{dt} = -\sum_n h_{mn}\phi_n\,, \quad \frac{d\phi_n^*}{dt} = \sum_m \phi_m^* h_{mn}\,, \tag{6.14}$$

where

$$h_{mn} = \sum_{ij} V_{mnij}\phi_i^*\phi_j + L_{mn}\,. \tag{6.15}$$

These equations can be interpreted as a Euclidean version of TDHE (2.16), (2.17). It should be stressed that the variables ϕ_n and ϕ_n^* are not complex conjugate in equations (6.14) in contrast to the standard TDHE.

C. Asymptotic conditions

In the vicinity of the "Hartree point" (6.8)–(6.11), the Hamilton equations for trajectories can be approximated by

$$\frac{dq_n}{dt} = \frac{\partial}{\partial p_n} H(p,q) = \varepsilon_0 q_n \,, \tag{6.16}$$

$$\frac{dp_n}{dt} = -\frac{\partial}{\partial q_n} H(p,q) \approx -\varepsilon_0 p_n \tag{6.17}$$

with the asymptotic solutions at $t \to -\infty$

$$q_n(t) \overset{t\to-\infty}{=} \delta_{n0} \exp\left[\varepsilon_0 \left(t-\tau\right)\right] , \tag{6.18}$$

$$p_n(t) \overset{t\to-\infty}{=} \delta_{n0} \left[-\varepsilon_0 \left(t-\tau\right)\right] . \tag{6.19}$$

According to Eq. (6.4) this asymptotic behavior corresponds to the Hartree energy E_0 (5.12)

$$H(p,q) = E_0 \,. \tag{6.20}$$

D. Integral of motion $\sum_n p_n q_n$

The Hamiltonian (6.4) and equations of motion (6.5) are invariant under the transformations

$$p_n \to \lambda^{-1} p_n \,, \quad q_n \to \lambda q_n \,. \tag{6.21}$$

Therefore

$$\left(p_n \frac{\partial}{\partial p_n} - q_n \frac{\partial}{\partial q_n} \right) H = 0 \,. \tag{6.22}$$

This identity can be rewritten in terms of the Poisson bracket

$$\left\{ \sum_n p_n q_n , H \right\} = 0 \,. \tag{6.23}$$

Thus we have the integral of motion

$$\sum_n p_n q_n = \text{const} \,. \tag{6.24}$$

For the trajectories with the asymptotic behavior (6.18), (6.19) we have

$$\sum_n p_n q_n = 1 \,. \tag{6.25}$$

E. Boundary condition

Using notation (6.1), (6.2), we can rewrite the asymptotic expansion (5.83) in the form

$$S\left(q\right) = \left. S(q^H) \right|_{q^H = \delta_{n0}} + \ln q_0 + \frac{1}{2} \sum_{m,n>0} X_{mn} \frac{q_m}{q_0} \frac{q_n}{q_0} + O\left[\left| \frac{q_{m>0}}{q_0} \right|^3 \right] . \tag{6.26}$$

This expression is valid for

$$|q_m| \ll q_0 \quad (m>0) \,. \tag{6.27}$$

Inserting this decomposition into Eq. (6.6), we find

$$p_n(q) = \frac{\partial S(q)}{\partial q_n} = \frac{1}{q_0} \delta_{n0} \quad \text{if} \quad |q_m| \ll q_0 \quad (m>0) \,. \tag{6.28}$$

This agrees with the asymptotic behavior of trajectories (6.18), (6.19) at $t \to -\infty$.

Now we understand that the action $S(q)$ is associated with the configuration of trajectories $p(t)$, $q(t)$ which start at $t \to -\infty$ at points p, q obeying Eqs. (6.8)–(6.10) with the asymptotic behavior (6.18), (6.19). All these trajectories have the same energy E_0 and the same integral of motion (6.25).

Let $p(t)$, $q(t)$ be one of these trajectories with energy E_0:

$$H\left[p(t), q(t)\right] = E_0\,. \tag{6.29}$$

Let us assume that at $t = 0$ this trajectory passes through the point $p^{(0)}, q^{(0)}$:

$$p(0) = p^{(0)}, \quad q(0) = q^{(0)}\,, \tag{6.30}$$

$$H\left(p^{(0)}, q^{(0)}\right) = E_0 \tag{6.31}$$

and at $t \to -\infty$ it has the asymptotic behavior (6.18), (6.19) with some

$$\tau = \tau(q^{(0)}) \tag{6.32}$$

depending on q_0:

$$q_n(t) \overset{t\to-\infty}{=} \delta_{n0} \exp\left\{\varepsilon_0 \left[t - \tau(q^{(0)})\right]\right\}\,, \tag{6.33}$$

$$p_n(t) \overset{t\to-\infty}{=} \delta_{n0} \left\{-\varepsilon_0 \left[t - \tau(q^{(0)})\right]\right\}\,. \tag{6.34}$$

Note that at $t \to -\infty$ we have along this trajectory

$$L = \sum_n p_n \dot{q}_n - H(p,q) \overset{t\to-\infty}{\to} \varepsilon_0 - E_0\,. \tag{6.35}$$

This means that the integral

$$\int_{-\infty}^{0} dt\,(L + E_0 - \varepsilon_0) \tag{6.36}$$

is convergent at $t \to -\infty$. Now let us define the function

$$S(q^{(0)}) = \int_{-\infty}^{0} dt\,(L + E_0 - \varepsilon_0) - \varepsilon_0 \tau(q^{(0)}) \tag{6.37}$$

and show that that this function obeys the differential equation (6.3).

According to Eq. (6.29) we have

$$L + E_0 = L + H = \sum_n p_n \dot{q}_n\,. \tag{6.38}$$

Therefore

$$S(q^{(0)}) = \int_{-\infty}^{0} dt \left[\sum_n p_n(t)\dot{q}_n(t) - \varepsilon_0\right] - \varepsilon_0 \tau(q^{(0)}) \tag{6.39}$$

Hence

$$\begin{aligned} S(q^{(0)}) &= \lim_{T\to-\infty} \int_{T+\tau(q^{(0)})}^{0} dt \sum_n p_n(t)\dot{q}_n(t) + \varepsilon_0 T \\ &= \lim_{T\to-\infty} \int_{\delta_{m0}\exp(\varepsilon_0 T)}^{q_m^{(0)}} \sum_n p_n dq_n + \varepsilon_0 T\,. \end{aligned} \tag{6.40}$$

Taking some large but fixed $|T|$ on the RHS, we see that this definition of $S(q^{(0)})$ has the standard property

$$\frac{\partial S(q^{(0)})}{\partial q_n^{(0)}} = p_n^{(0)} \,, \tag{6.41}$$

where $p_n^{(0)}$ is the momentum $p_n(t)$ taken at $t = 0$ according to Eq. (6.30). Inserting Eq. (6.41) into Eq. (6.31) and taking into account that the point $q^{(0)}$ is arbitrary, we find that the function $S(q)$ defined by Eq. (6.37) obeys the Hamilton–Jacobi equation (6.3).

Since the definition (6.37) of $S(q)$ is based on the trajectories with the asymptotic behavior (6.33), (6.34), we automatically have the property (6.28). Therefore function $S(q)$ defined by Eq. (6.37) also obeys the boundary condition (6.26). Since both differential equation (6.3) and the boundary condition (6.26) are satisfied, we conclude that Eq. (6.37) gives a correct representation for the functional

$$W(g) \equiv S(q^{(0)}) \,, \quad g \equiv q^{(0)} \tag{6.42}$$

which determines the large-N_c behavior (1.7).

The construction described in this section leads us to the following algorithm of the calculation of $W(g)$.

1) Solve equations of motion (6.5) imposing boundary condition

$$q(0) = g \tag{6.43}$$

with given g (6.60) and boundary conditions (6.33), (6.34)

$$q_n(t) \stackrel{t\to-\infty}{=} \delta_{n0} \exp\left\{\varepsilon_0 \left[t - \tau(g)\right]\right\} \,, \tag{6.44}$$

$$p_n(t) \stackrel{t\to-\infty}{=} \delta_{n0} \left\{-\varepsilon_0 \left[t - \tau(g)\right]\right\} \,. \tag{6.45}$$

with *unknown* $\tau(g)$. In this way one finds the trajectory and $\tau(g)$.

2) Now $W(g)$ is given by Eqs. (6.39), (6.42):

$$W(g) = \int_{-\infty}^{0} dt \left[\sum_n p_n(t)\dot{q}_n(t) - \varepsilon_0\right] - \varepsilon_0 \tau(g) \,. \tag{6.46}$$

F. Case $L_{mn} = 0$

Let us consider the special case when

$$L_{mn} = 0 \tag{6.47}$$

in the Hamiltonian (6.4). In this case the Hartree equations (5.10), (5.12) lead to the simple relation

$$E_0 = \frac{1}{2}\varepsilon_0 \,. \tag{6.48}$$

In the case (6.47) the Hamiltonian (6.4) is quadratic in p so that

$$\sum_n p_n \frac{\partial H(p,q)}{\partial p_n} = 2H(p,q) \,. \tag{6.49}$$

Combining this with Eq. (6.29), we find

$$\sum_n p_n \dot{q}_n = \sum_n p_n \frac{\partial H}{\partial p_n} = 2H(p,q) = 2E_0 \,. \tag{6.50}$$

Taking into account Eq. (6.48), we arrive at

$$\sum_n p_n \dot{q}_n - \varepsilon_0 = 2E_0 - \varepsilon_0 = 0 \,. \tag{6.51}$$

Now we insert this into Eq. (6.39)

$$S(q^{(0)}) = -\varepsilon_0 \tau(q^{(0)})\,. \tag{6.52}$$

This allows us to rewrite Eqs. (6.33) and (6.34) in the form

$$q_n(t) \overset{t\to-\infty}{=} \delta_{n0} \exp\left[\varepsilon_0 t + S(q^{(0)})\right]\,, \tag{6.53}$$

$$p_n(t) \overset{t\to-\infty}{=} \delta_{n0} \left[-\varepsilon_0 t - S(q^{(0)})\right]\,. \tag{6.54}$$

Thus in the case (6.47) the action can be directly read from the asymptotic behavior of $q_n(t)$, $p_n(t)$ at $t \to -\infty$. Remember that these trajectories are fixed by extra conditions (6.30)

$$q(0) = q^{(0)} \tag{6.55}$$

and (6.20), (6.48)

$$H\left(p^{(0)}, q^{(0)}\right) = E_0 = \frac{1}{2}\varepsilon_0\,. \tag{6.56}$$

Now we conclude from Eqs. (6.42), (6.53), (6.54) that in the case $L_{mn} = 0$ the functional $W(g)$ is given by the equation

$$W(g) = \ln I(g) \quad (\text{if } L_{mn} = 0)\,, \tag{6.57}$$

where $I(g)$ is the parameter of the trajectories obeying the asymptotic conditions

$$q_n(t) \overset{t\to-\infty}{=} \delta_{n0} I(g) e^{\varepsilon_0 t}\,, \tag{6.58}$$

$$p_n(t) \overset{t\to-\infty}{=} \frac{\delta_{n0}}{I(g)} e^{-\varepsilon_0 t}\,, \tag{6.59}$$

$$q_n(0) = g_n\,. \tag{6.60}$$

Let us summarize. In the case $L_{mn} = 0$ the calculation of the functional $W(g)$ reduces to the following steps:

1) Solve equations of motion (6.5) imposing boundary conditions (6.58) – (6.60) with given g and with *unknown* $I(g)$. In this way one finds $I(g)$.

2) Express $W(g)$ via $I(g)$ according to Eq. (6.57).

VII. EXAMPLE: ASYMMETRIC ROTATOR AS A LARGE-N_c SYSTEM

A. Model

In the above sections we have considered the general theory of fermionic systems described by operators a_{nc}, a^+_{nc} (2.3) with $c = 1, 2, \ldots, N_c$. Now we want to consider the simplest version of these models with only two values for the index n of a_{nc}. It is convenient to interpret this two-valued index as the projection of spin 1/2. In other words, we want to consider the model described by the Hamiltonian

$$H = \frac{1}{2N_c} \sum_{ab} K_{ab} J_a J_b + \sum_a Y_a J_a\,, \tag{7.1}$$

where

$$J_b = \sum_{j,j'=\pm\frac{1}{2}} \sum_{c=1}^{N_c} \frac{1}{2} a^+_{j'c} \left(\tau_b\right)_{j'j} a_{jc}\,. \tag{7.2}$$

Obviously this Hamiltonian belongs to the class (2.8) with parameters

$$V_{n_1 n_2 n_3 n_4} = \frac{1}{4} \sum_{ab} K_{ab} \left(\tau_a\right)_{n_1 n_2} \left(\tau_b\right)_{n_3 n_4}\,, \tag{7.3}$$

$$L_{n_1 n_2} = \frac{1}{2} \sum_a Y_a \left(\tau_a\right)_{n_1 n_2}\,. \tag{7.4}$$

On the other hand, Hamiltonian (7.1) is nothing else but the rotator described by the angular momentum J_a

$$[J_a, J_b] = i\varepsilon_{abc} J_c \,. \tag{7.5}$$

We will consider the case of the asymmetric rotator ($I_{ab} \neq I\delta_{ab}$) but the quantum number $\mathbf{J}^2$ is still conserved:

$$\left[\mathbf{J}^2, H\right] = 0 \tag{7.6}$$

and has the standard eigenvalues

$$\mathbf{J}^2 = J(J+1) \,. \tag{7.7}$$

We want to study the states made of N_c quarks and antisymmetric in color. The Fermi statistics and the color antisymmetry lead to the complete symmetry of the spin wave function so that we deal with

$$J = \frac{N_c}{2} \,. \tag{7.8}$$

Thus the algebraic formulation of the problem of the baryon is obvious: take the sector with $J = N_c/2$ and diagonalize the Hamiltonian (7.1) in this sector.

B. Direct semiclassical approach

In principle, we can find the spectrum of low-lying states of the Hamiltonian (7.1) at large N_c solving the general Hartree equation (4.8) and diagonalizing the RPA Hamiltonian (5.33). However, the same results can be obtained by directly applying the semiclassical approximation to the large momentum J (7.8) in the Hamiltonian (7.1).

Instead of solving the Hartree equation (4.8) we can simply minimize the Hamiltonian (7.1) considering it as a function of the classical momentum $\mathbf{J}$. This minimization should be performed assuming the classical version of the constraint (7.8). Introducing the Lagrange multiplier λ, we arrive at the extremum problem for the ground state energy (4.5) in the leading order of the $1/N_c$ expansion:

$$N_c E_0 = \min_{\mathbf{J}} \left[\frac{1}{2N_c} \sum_{ab} \left(K_{ab} - \lambda\delta_{ab}\right) J_a J_b + \sum_a Y_a J_a \right] , \tag{7.9}$$

This leads to the equation

$$\frac{1}{N_c} \sum_b \left(K_{ab} - \lambda\delta_{ab}\right) J_b + Y_a = 0 \,. \tag{7.10}$$

If $K - \lambda$ is not degenerate, then

$$J_a = -N_c \sum_b \left[(K-\lambda)^{-1}\right]_{ab} Y_b \,. \tag{7.11}$$

Inserting this into Eq. (7.8), we find

$$\left(Y, (K-\lambda)^{-2} Y\right) = \frac{1}{4} \,. \tag{7.12}$$

This equation determines λ. Knowing λ, we can find J from Eq. (7.11).

The leading order of the $1/N_c$ expansion (4.5) for the energy of the lowest states is determined by the parameter

$$\begin{aligned} E_0 &= \frac{1}{N_c}\left[\frac{1}{2N_c} K_{ab} J_a J_b + Y_a J_a\right] = \frac{1}{2} Y (K-\lambda)^{-1} K (K-\lambda)^{-1} Y - Y (K-\lambda)^{-1} Y \\ &= -\frac{1}{2} Y \frac{K - 2\lambda}{(K-\lambda)^2} Y \,. \end{aligned} \tag{7.13}$$

In principle, we may also have other solutions of Eq. (7.10) corresponding to

$$\det(K - \lambda) = 0 \,. \tag{7.14}$$

C. Special case

Below we concentrate on the case when matrix K_{ab} is diagonal

$$K_{ab} = K_a \delta_{ab} \tag{7.15}$$

and

$$Y_1 = Y_2 = 0\,. \tag{7.16}$$

We also assume that

$$Y_3 > 0\,, \tag{7.17}$$

$$K_1 - K_3 + 2Y_3 > 0\,, \tag{7.18}$$

$$K_2 - K_3 + 2Y_3 > 0\,. \tag{7.19}$$

In this case Eq. (7.12) yields

$$K_3 - \lambda = \pm 2Y_3 \tag{7.20}$$

and Eq. (7.13) results in

$$E_0 = -\frac{1}{8}\left(K_3 - 2\lambda\right) = -\frac{1}{8}\left[K_3 - 2\left(K_3 \mp 2Y_3\right)\right]\,. \tag{7.21}$$

According to Eq. (7.17) the minimal energy corresponds to the upper sign. Thus

$$E_0 = \frac{1}{8}\left(K_3 - 4Y_3\right)\,. \tag{7.22}$$

We find from Eq. (7.11)

$$J_1 = J_2 = 0\,, \quad J_3 = -\frac{N_c}{2}\,. \tag{7.23}$$

One can show that under conditions (7.17)–(7.19) this solution gives the true semiclassical ground state and solutions of Eq. (7.14) can be ignored.

D. Hartree equation

The semiclassical spin J_a has the following interpretation in terms of the solution of the Hartree equation (4.8)

$$J_a = \frac{N_c}{2}\sum_{mn}\phi^*_m\left(\tau_a\right)_{mn}\phi_m\,. \tag{7.24}$$

Therefore the classical solution (7.23) corresponds to

$$\phi_m = \delta_{m2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{7.25}$$

In the Hartree equation (4.8) we have the single-particle Hamiltonian

$$h_{n_1 n_2} = \sum_{n_3 n_4} V_{n_1 n_2 n_3 n_4}\phi^*_{n_3}\phi_{n_4} + L_{n_1 n_2} = \frac{1}{4}\left(-K_3 + 2Y_3\right)\left(\tau_3\right)_{n_1 n_2}\,. \tag{7.26}$$

Hence the single-particle energy ε_0 is

$$\varepsilon_0 = \frac{1}{4}\left(K_3 - 2Y_3\right)\,. \tag{7.27}$$

It is easy to see that the general expression for the Hartree energy E_0 (4.10) reproduces the above result (7.22). The single nonoccupied Hartree state is

$$\phi^{(1)}_m = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.28}$$

and its energy is

$$\varepsilon_1 = -\varepsilon_0 = -\frac{1}{4}\left(K_3 - 2Y_3\right)\,. \tag{7.29}$$

E. RPA equation

In the general RPA matrix $\mathcal{R}$ (5.34), the indices m, n correspond to the nonoccupied eigenstates of the Hartree equation. In our case we have only one nonoccupied state (7.28). We label the occupied Hartree state with $m = 0$ whereas the nonoccupied state is labeled by $m = 1$. As a result, matrix $\mathcal{R}$ (5.34) takes the form

$$\mathcal{R} = \begin{pmatrix} V_{0101} & V_{1001} + \varepsilon_1 - \varepsilon_0 \\ V_{1001} + \varepsilon_1 - \varepsilon_0 & V_{1010} \end{pmatrix} . \tag{7.30}$$

Using Eq. (7.3), we find for the diagonal tensor K_{ab} (7.15)

$$V_{0101} = V_{1010} = \frac{1}{4} (K_1 - K_2) , \tag{7.31}$$

$$V_{1001} = \frac{1}{4} (K_1 + K_2) . \tag{7.32}$$

With these coefficients $V_{n_1 n_2 n_3 n_4}$ and with expression (7.29) for $\varepsilon_1 - \varepsilon_0$, we obtain

$$\mathcal{R} = \frac{1}{4} \begin{pmatrix} K_1 - K_2 & K_1 + K_2 + 2(-K_3 + 2Y_3) \\ K_1 + K_2 + 2(-K_3 + 2Y_3) & K_1 - K_2 \end{pmatrix} . \tag{7.33}$$

This leads to the Hamiltonian (5.33)

$$H_{\mathrm{RPA}} = \frac{1}{8} \left[(K_1 - K_2) \left(aa + a^+ a^+ \right) + \left[K_1 + K_2 + 2(-K_3 + 2Y_3) \right] \left(a^+ a + aa^+ \right) \right] - \frac{1}{4} (-K_3 + 2Y_3) . \tag{7.34}$$

Performing Bogolyubov transformation

$$a = \frac{\kappa^{1/2} + \kappa^{-1/2}}{2} b + \frac{\kappa^{1/2} - \kappa^{-1/2}}{2} b^+ \tag{7.35}$$

with

$$\kappa = \sqrt{\frac{K_2 - K_3 + 2Y_3}{K_1 - K_3 + 2Y_3}} , \tag{7.36}$$

we find

$$H_{\mathrm{RPA}} = \frac{\omega}{2} \left(b^+ b + b b^+ \right) + \frac{1}{4} (K_3 - 2Y_3) , \tag{7.37}$$

where

$$\omega = \frac{1}{2} \sqrt{(K_1 - K_3 + 2Y_3)(K_2 - K_3 + 2Y_3)} . \tag{7.38}$$

According to Eqs. (7.36) and (7.38) we have

$$K_1 = K_3 - 2Y_3 + 2\kappa^{-1} \omega , \tag{7.39}$$

$$K_2 = K_3 - 2Y_3 + 2\kappa\omega . \tag{7.40}$$

Comparing Eq. (5.49) with Eq. (7.35), we see that

$$\mathcal{S} = \frac{1}{2} \begin{pmatrix} \kappa^{1/2} + \kappa^{-1/2} & \kappa^{1/2} - \kappa^{-1/2} \\ \kappa^{1/2} - \kappa^{-1/2} & \kappa^{1/2} + \kappa^{-1/2} \end{pmatrix} \tag{7.41}$$

and the only RPA excitation energy is

$$\Omega_1 = \omega . \tag{7.42}$$

Comparing our result (7.41) for $\mathcal{S}$ with Eq. (5.45), we obtain

$$\alpha_{11} = \kappa^{1/2} + \kappa^{-1/2} , \tag{7.43}$$

$$\beta_{11} = \kappa^{1/2} - \kappa^{-1/2} . \tag{7.44}$$

Using Eq. (5.60), we find

$$X_{11} = \left[\beta \left(\alpha^* \right)^{-1} \right]_{11} = \frac{\kappa^{1/2} - \kappa^{-1/2}}{\kappa^{1/2} + \kappa^{-1/2}} . \tag{7.45}$$

F. Method of generating functionals

In the case of the Hamiltonian (7.1), Eq. (4.6) for the functional $W(g)$ takes the form

$$\sum_{ab} \frac{1}{2} K_{ab} \mathcal{J}_a \mathcal{J}_b + \sum_a Y_a \mathcal{J}_a = E_0 \,, \tag{7.46}$$

where

$$\mathcal{J}_a = g_i \left(\frac{\tau_a}{2} \right)_{ij} \frac{\partial W(g)}{\partial g_j} \,. \tag{7.47}$$

According to Eq. (4.1) function $W(g_1, g_2)$ has a nontrivial dependence only on the ration g_1/g_2 so that we can represent $W(g_1, g_2)$ in the form

$$W(g_1, g_2) = f\left(\frac{g_1}{g_2} \right) + \frac{1}{2} \ln (g_1 g_2) \tag{7.48}$$

with some function f. It is easy to see that Eq. (7.46) leads to an ordinary differential equation for the function f. In order to solve this equation it is convenient to change the variables:

$$\zeta = \left(\frac{g_1}{g_2} \right)^2 , \quad w = \ln (g_1 g_2) \,, \quad R = 4f \,. \tag{7.49}$$

In terms of these variables representation (7.48) becomes

$$W(g_1, g_2) = \frac{1}{4} R(\zeta) + \frac{1}{2} w \,. \tag{7.50}$$

A straightforward calculation of expressions (7.47) shows that

$$\mathcal{J}_1 = \frac{1}{4\sqrt{\zeta}} \left[(1+\zeta) - (\zeta - 1) \frac{dR(\zeta)}{d\zeta} \right] , \tag{7.51}$$

$$\mathcal{J}_2 = \frac{i}{4\sqrt{\zeta}} \left[(1-\zeta) + (\zeta + 1) \frac{dR(\zeta)}{d\zeta} \right] , \tag{7.52}$$

$$\mathcal{J}_3 = \frac{1}{2} \zeta \frac{\partial R(\zeta)}{\partial \zeta} \,. \tag{7.53}$$

In the case (7.15), (7.16) we find from Eq. (7.46)

$$\frac{1}{2} \sum_{a=1}^{3} K_a \left(\mathcal{J}_a \right)^2 + Y_3 \mathcal{J}_3 = E_0 \,. \tag{7.54}$$

Inserting Eqs. (7.51)–(7.53), we obtain

$$\frac{K_1}{2\zeta} \left[(1+\zeta) - (\zeta - 1) \zeta \frac{dR(\zeta)}{d\zeta} \right]^2 - \frac{K_2}{2\zeta} \left[(1-\zeta) + (\zeta + 1) \zeta \frac{dR(\zeta)}{d\zeta} \right]^2$$
$$+ 2K_3 \left[\zeta \frac{dR(\zeta)}{d\zeta} \right]^2 + 8 Y_3 \zeta \frac{dR(\zeta)}{d\zeta} = 16 E_0 \,. \tag{7.55}$$

G. Agreement with the Hartree equation

At the point $g_n = \phi_n^*$ (4.11) corresponding to the solution of the Hartree equation we must have according to Eq. (7.48)

$$\phi_n = \frac{\partial}{\partial g_n} \left[\frac{1}{4} R \left(\left(\frac{g_1}{g_2} \right)^2 \right) + \frac{1}{2} \ln (g_1 g_2) \right] \,. \tag{7.56}$$

Taking into account Eq. (7.25), we see that this imposes the following constraint on the behavior of $R(\zeta)$ at small ζ

$$R(\zeta) \overset{\zeta\to 0}{=} \text{const} - \ln\zeta + \dots \tag{7.57}$$

Let us find the next term of this small ζ expansion. Applying Eq. (5.83) to our case and taking X_{11} from Eq. (7.45), we obtain

$$W(g) = W(\phi) + \ln g_2 + \frac{1}{2}\frac{\kappa^{1/2}-\kappa^{-1/2}}{\kappa^{1/2}+\kappa^{-1/2}}\left(\frac{g_1}{g_2}\right)^2 + O\left(\left(\frac{g_1}{g_2}\right)^4\right). \tag{7.58}$$

According to Eq. (4.22) we have for normalizable states (with the appropriately chosen phase)

$$W(\phi) = 0\,. \tag{7.59}$$

Therefore

$$W(g) = \ln g_2 + \frac{1}{2}\frac{\kappa^{1/2}-\kappa^{-1/2}}{\kappa^{1/2}+\kappa^{-1/2}}\left(\frac{g_1}{g_2}\right)^2 + O\left(\left(\frac{g_1}{g_2}\right)^4\right). \tag{7.60}$$

Combining this with Eq. (7.50), we find

$$R(\zeta) \overset{\zeta\to 0}{=} -\ln\zeta + 2\frac{\kappa^{1/2}-\kappa^{-1/2}}{\kappa^{1/2}+\kappa^{-1/2}}\zeta + O\left(\zeta^2\right). \tag{7.61}$$

H. Calculation of $W(g)$

Eq. (7.55) is a quadratic equation with respect to $\partial R/\partial z$ which can be easily solved. Using expression (7.22) for E_0, we find from Eq. (7.55)

$$\zeta\frac{\partial R}{\partial\zeta} = \frac{-\left[\eta(1-\zeta^2)+C\zeta\right] \pm 2\zeta\sqrt{(\xi^2-\eta^2)-\eta C\zeta}}{\eta(1+\zeta^2)-(2\xi-C)\,\zeta}\,. \tag{7.62}$$

Here we have introduced the compact notation

$$C = 4Y_3\,, \tag{7.63}$$

$$\xi = \left(\kappa^{-1}+\kappa\right)\omega = \frac{1}{2}(K_1+K_2) - K_3 + 2Y_3\,, \tag{7.64}$$

$$\eta = \left(\kappa^{-1}-\kappa\right)\omega = \frac{1}{2}(K_1-K_2)\,. \tag{7.65}$$

In order to fix the sign uncertainty on the RHS of (7.62), let us consider the limit $\zeta\to 0$

$$\zeta\frac{\partial R}{\partial\zeta} \overset{\zeta\to 0}{=} -1 + \frac{2\zeta}{\eta}\left(-\xi \pm \sqrt{\xi^2-\eta^2}\right) = -1 + 2\zeta\frac{-\left(\kappa^{-1}+\kappa\right)\pm 2}{\kappa^{-1}-\kappa}\,, \tag{7.66}$$

On the other hand, we find from Eq. (7.61)

$$\zeta\frac{\partial R}{\partial\zeta} \overset{\zeta\to 0}{=} -1 + 2\frac{\kappa^{1/2}-\kappa^{-1/2}}{\kappa^{1/2}+\kappa^{-1/2}}\zeta + O\left(\zeta^{3/2}\right). \tag{7.67}$$

Comparing these two expressions, we conclude that we must have

$$\frac{\kappa^{1/2}-\kappa^{-1/2}}{\kappa^{1/2}+\kappa^{-1/2}} = \frac{-\left(\kappa^{-1}+\kappa\right)\pm 2}{\kappa^{-1}-\kappa}\,. \tag{7.68}$$

This identity holds if we choose the upper sign in the numerator of the RHS. Now Eq. (7.62) takes the form

$$\frac{\partial R}{\partial\zeta} = -\frac{1}{\zeta} + 2\frac{\eta\zeta-\xi+\sqrt{\xi^2-\eta^2-\eta C\zeta}}{\eta(1+\zeta^2)-(2\xi-C)\,\zeta}\,. \tag{7.69}$$

One can easily integrate this equation. The integration constant is fixed by Eq. (7.61). The calculation of the integral yields

$$R(\zeta) = -\ln\zeta + F(\zeta) - F(0)\,, \tag{7.70}$$

where

$$F(\zeta) = 4\frac{\nu_1 \ln\left[u(\zeta)-\nu_1\right] - \nu_2 \ln\left[u(\zeta)-\nu_2\right]}{\nu_1-\nu_2}\,, \tag{7.71}$$

$$\nu_{1,2} = \frac{-4Y_3 \pm \sqrt{(K_1-K_3)(K_2-K_3)}}{|K_1-K_2|}\,, \tag{7.72}$$

$$u(\zeta) = \frac{2\sqrt{(K_1-K_3+2Y_3)\,(K_2-K_3+2Y_3) - 2Y_3(K_1-K_2)\zeta}}{|K_1-K_2|}\,. \tag{7.73}$$

Now we insert Eqs. (7.70) and (7.49) into Eq. (7.50)

$$W(g_1,g_2) = \ln g_2 + \frac{1}{4}F\left(\left(\frac{g_1}{g_2}\right)^2\right) - \frac{1}{4}F(0)\,. \tag{7.74}$$

Thus we have computed function $W(g_1,g_2)$ which determines the exponential part of the large-N_c behavior (1.7) for the system with the Hamiltonian (7.1). One should keep in mind that function $F(\zeta)$ has branch points. At small ζ this function is regular. According to Eq. (7.61)

$$F(\zeta) \overset{\zeta\to 0}{=} F(0) + 2\frac{\kappa^{1/2}-\kappa^{-1/2}}{\kappa^{1/2}+\kappa^{-1/2}}\zeta + O\left(\zeta^2\right)\,. \tag{7.75}$$

Therefore for small $|g_1/g_2|$ our result for $W(g_1,g_2)$ is unambiguous. However, at larger $|g_1/g_2|$ one can meet cuts. In this case the determination of the relevant branch of $W(g_1,g_2)$ will require a special analysis.

VIII. MODELS WITH THE NONTRIVIAL VACUUM

A. Color singlet states with MN_c quarks

Our previous work was devoted to models of the baryon wave function based on systems with the trivial vacuum. In such systems the baryon wave function comes directly from the solution of the Schrödinger equation for the baryon (2.2). In other words, we worked with models which have quarks but no antiquarks. Within this class of models we could check the general properties of the large-N_c limit like the exponential large-N_c behavior (1.7) and the factorization of the preexponential factors A_B (1.18).

Now we want to show that these general large-N_c properties also hold in more complicated models containing antiquark degrees of freedom in addition to quarks. It is convenient to formulate these models in terms of the old Dirac picture when both vacuum and baryons represented as a result of the occupation of the bare vacuum with quarks. In these models the physical vacuum contains MN_c quarks whereas the baryon is made of $(M+1)N_c$ quarks where M is some integer number. This allows us to define the quark wave function of baryons via the transition matrix element (2.5).

We want to start from the case $M=1$ in order to concentrate on the idea of the method and to avoid unnecessary complications. The generalization for arbitrary M is discussed in Sec. VIII J in terms of the Schrödinger approach. In Secs. IX, X the case of arbitrary M will be analyzed using the path integral method. It should be stressed that the case $M=1$ from which we want to start is rather nonphysical: the vacuum made of N_c quarks is a fermion for odd N_c whereas the baryon made of $2N_c$ quarks is always a boson. Nevertheless the $M=1$ case is a good starting point for understanding the dynamics of models including quark-antiquark pairs.

B. Color singlet states with $2N_c$ quarks

The color singlet states with $2N_c$ quarks above the *bare vacuum* $|\Omega\rangle$ can be represented in the form

$$|B\rangle = \sum_{i_k j_k} B^{i_1 i_2 \dots i_{N_c}}_{j_1 j_2 \dots j_{N_c}} \prod_{c=1}^{N_c} a^+_{i_c c} \prod_{c'=1}^{N_c} a^+_{j_{c'} c'} |\Omega\rangle \,. \tag{8.1}$$

Without any loss of generality we can assume that $B^{i_1 i_2 \dots i_{N_c}}_{j_1 j_2 \dots j_{N_c}}$ is symmetric under permutations of upper indices $i_1 i_2 \dots i_{N_c}$ as well as under permutations of lower indices $j_1 j_2 \dots j_{N_c}$:

$$B^{i_1 i_2 \dots i_{N_c}}_{j_1 j_2 \dots j_{N_c}} = B^{\{i_1 i_2 \dots i_{N_c}\}}_{\{j_1 j_2 \dots j_{N_c}\}} \,. \tag{8.2}$$

The Fermi statistics of quarks

$$a^+_{i_k c} a^+_{j_k c'} = -a^+_{j_k c'} a^+_{i_k c} \tag{8.3}$$

allows us to antisymmetrize $B^{i_1 i_2 \dots i_{N_c}}_{j_1 j_2 \dots j_{N_c}}$ with respect to $i_1 \leftrightarrow j_1$:

$$B^{i_1 i_2 \dots i_{N_c}}_{j_1 j_2 \dots j_{N_c}} \to \frac{1}{2} \left(B^{i_1 i_2 \dots i_{N_c}}_{j_1 j_2 \dots j_{N_c}} - B^{j_1 i_2 \dots i_{N_c}}_{i_1 j_2 \dots j_{N_c}} \right) . \tag{8.4}$$

We can continue this process antisymmetrizing in all pairs $i_k \leftrightarrow j_k$. After that the original symmetry in $\{i_1 i_2 \dots i_{N_c}\}$ and in $\{j_1 j_2 \dots j_{N_c}\}$ is lost but we still have the symmetry with respect to the permutations of any pair $\{i_m j_m\}$ with another pair $\{i_n j_n\}$.

Note that our construction of tensors $B^{i_1 i_2 \dots i_{N_c}}_{j_1 j_2 \dots j_{N_c}}$ based on
1) symmetrization in $\{i_1 i_2 \dots i_{N_c}\}$,
2) symmetrization in $\{j_1 j_2 \dots j_{N_c}\}$,
3) antisymmetrizations $[i_1 j_1]$, $[i_2 j_2]$,..., $[i_{N_c} j_{N_c}]$
coincides with the construction of tensors associated with the rectangular Young tableau containing two rows and N_c columns.

Assuming that the tensor $B^{i_1 i_2 \dots i_{N_c}}_{j_1 j_2 \dots j_{N_c}}$ obeys the above symmetry properties, let us construct the "generating functional"

$$\tilde{\Phi}_B(\gamma) = \sum_{i_k j_k} B^{i_1 i_2 \dots i_{N_c}}_{j_1 j_2 \dots j_{N_c}} \gamma_{i_1 j_1} \gamma_{i_2 j_2} \dots \gamma_{i_{N_c} j_{N_c}} \,, \tag{8.5}$$

depending on the antisymmetric matrix "source"

$$\gamma_{ij} = -\gamma_{ji} \,. \tag{8.6}$$

This construction generalizes the case of states with N_c quarks (3.3). We use the tilded notation $\tilde{\Phi}$ in order to distinguish the functional $\tilde{\Phi}_B(\gamma)$ depending on the antisymmetric tensors γ_{ij} from the functional $\tilde{\Phi}_B(g)$ depending on "vectors" g_i. According to Eqs. (8.1), (8.5) the functional $\tilde{\Phi}_B(\gamma)$ corresponds to the transition matrix element between the baryon $|B\rangle$ and the *bare* vacuum $|\Omega\rangle$. On the other hand, the functional $\Phi_B(g)$ is associated with the transition matrix element (2.6) between the baryon $|B\rangle$ and the *physical* vacuum $|0\rangle$.

Now we have the correspondence between the states $|B\rangle$ (8.1) and functions $\tilde{\Phi}_B(\gamma)$ (8.5):

$$|B\rangle \to \tilde{\Phi}_B(\gamma) \tag{8.7}$$

Within this correspondence operators T_{mn} (3.5) are mapped to

$$T_{mn} = \sum_{c=1}^{N_c} a^+_{mc} a_{nc} \quad \to \quad \sum_j \gamma_{mj} \frac{\partial}{\partial \gamma_{nj}} \,. \tag{8.8}$$

According to Eq. (8.6) tensors γ_{nj} are antisymmetric. Therefore we must be careful about the normalization of the derivative $\partial / \partial \gamma_{nj}$. Our choice is

$$\frac{\partial}{\partial \gamma_{pq}} \gamma_{mn} = \delta_{mp} \delta_{nq} - \delta_{np} \delta_{mq} \,. \tag{8.9}$$

Obviously we have

$$\sum_{ij}\gamma_{ij}\frac{\partial}{\partial\gamma_{ij}}\tilde{\Phi}_B(\gamma)=2N_c\tilde{\Phi}_B(\gamma)\,. \tag{8.10}$$

Now we turn to the Schrödinger equation. For simplicity we will assume that

$$L_{mn}=0 \tag{8.11}$$

in the Hamiltonian (2.8). Using the γ representation (8.8) for this Hamiltonian, we can rewrite the Schrödinger equation

$$H\tilde{\Phi}_B(\gamma)=\mathcal{E}_B\tilde{\Phi}_B(\gamma) \tag{8.12}$$

in the form

$$\frac{1}{2N_c}\sum_{mnpq}V_{mnpq}\left(\sum_j\gamma_{mj}\frac{\partial}{\partial\gamma_{nj}}\right)\left(\sum_k\gamma_{pk}\frac{\partial}{\partial\gamma_{qk}}\right)\tilde{\Phi}_B(\gamma)=\mathcal{E}_B\tilde{\Phi}_B(\gamma)\,. \tag{8.13}$$

C. Large-N_c limit

At large N_c we use the standard ansatz for the generating function corresponding to the $2N_c$ quark state:

$$\tilde{\Phi}_B(\gamma)=N_c^{\nu_B}\tilde{A}_B(\gamma)\exp\left[N_cW_{\rm bar}(\gamma)\right]\,. \tag{8.14}$$

Inserting this ansatz into Eq. (8.10), we find

$$\sum_{ij}\gamma_{ij}\frac{\partial}{\partial\gamma_{ij}}W_{\rm bar}(\gamma)=2\,. \tag{8.15}$$

Combining the large-N_c expression (8.14) with the Schrödinger equation (8.13) and using the $1/N_c$ expansion for the energy

$$\mathcal{E}_B=N_cE_0+\Delta E_B+O(N_c^{-1})\,, \tag{8.16}$$

we find in the leading order

$$\frac{1}{2}\sum_{mnpq}V_{mnpq}\mathcal{T}_{mn}(\gamma)\mathcal{T}_{pq}(\gamma)=E_0\,, \tag{8.17}$$

where

$$\mathcal{T}_{mn}(\gamma)\equiv\sum_j\gamma_{mj}\frac{\partial W_{\rm bar}(\gamma)}{\partial\gamma_{nj}}\,. \tag{8.18}$$

In the next-to-leading order we obtain

$$\sum_{mnpq}\left[V_{mnpq}\mathcal{T}_{mn}(\gamma)\sum_k\gamma_{pk}\frac{\partial\ln\tilde{A}_B(\gamma)}{\partial\gamma_{qk}}+\frac{1}{2}V_{mnpq}\sum_j\gamma_{mj}\frac{\partial\mathcal{T}_{pq}(\gamma)}{\partial\gamma_{nj}}\right]=\Delta E_B\,. \tag{8.19}$$

D. Next-to-leading order

In the NLO we have equation (8.19). The structure of this equation allows for the solutions of the form

$$\tilde{A}_B(\gamma)=\tilde{A}^{(0)}(\gamma)\prod_\alpha\left[\xi_\alpha(\gamma)\right]^{n_\alpha}\,, \tag{8.20}$$

$$\Delta E_B = \Delta E_0 + \sum_\alpha n_\alpha \Delta E_\alpha \,, \tag{8.21}$$

where $\tilde{A}^{(0)}(\gamma)$ obeys the inhomogeneous equation (8.19)

$$\sum_{mnpq} \left[V_{mnpq} \mathcal{T}_{mn}(\gamma) \sum_k \gamma_{pk} \frac{\partial \ln \tilde{A}^{(0)}(\gamma)}{\partial \gamma_{qk}} + \frac{1}{2} V_{mnpq} \sum_j \gamma_{mj} \frac{\partial \mathcal{T}_{pq}(\gamma)}{\partial \gamma_{nj}} \right] = \Delta E_0 \,. \tag{8.22}$$

and $\xi_k(\gamma)$ are solutions of the homogeneous equation

$$\sum_{mnpq} V_{mnpq} \mathcal{T}_{mn}(\gamma) \left[\sum_k \gamma_{pk} \frac{\partial \ln \xi_\alpha(\gamma)}{\partial \gamma_{qk}} \right] = \Delta E_\alpha \,. \tag{8.23}$$

E. Overlap matrix elements

Now let us consider two baryon states $|B_1\rangle$ and $|B_2\rangle$. Let us calculate the overlap matrix element for these two states

$$\begin{aligned}
\langle B_1 | B_2 \rangle &= 2^{N_c} \sum_{i_k j_k} \left[(B_1)^{i_1 i_2 \ldots i_{N_c}}_{j_1 j_2 \ldots j_{N_c}} \right]^* (B_2)^{i_1 i_2 \ldots i_{N_c}}_{j_1 j_2 \ldots j_{N_c}} \\
&= \frac{(2N_c)^{N_c}}{N_c!} \frac{\int d\gamma d\gamma^* \exp\left(-N_c \sum\limits_{ij} \gamma_{ij} \gamma^*_{ij} \right) \left[\tilde{\Phi}_{B_1}(\gamma) \right]^* \left[\tilde{\Phi}_{B_2}(\gamma) \right]}{\int d\gamma d\gamma^* \exp\left(-N_c \sum\limits_{ij} \gamma_{ij} \gamma^*_{ij} \right)} \\
&\to \frac{(2N_c)^{N_c}}{N_c!} \frac{\int d\gamma d\gamma^* \exp\left\{ -N_c \left\{ \sum\limits_{ij} \gamma_{ij} \gamma^*_{ij} + [W_{\rm bar}(\gamma)]^* + W_{\rm bar}(\gamma) \right\} \right\} \left[\tilde{A}_{B_1}(\gamma) \right]^* \left[\tilde{A}_{B_2}(\gamma) \right]}{\int d\gamma d\gamma^* \exp\left(-N_c \sum\limits_{ij} \gamma_{ij} \gamma^*_{ij} \right)} \,.
\end{aligned} \tag{8.24}$$

At large N_c the integral over γ can be computed using the saddle method. The saddle point equation is

$$2\gamma^*_{ij} = - \left. \frac{\partial W_{\rm bar}(\gamma)}{\partial \gamma_{ij}} \right|_{\gamma = \gamma^{(0)}} \,. \tag{8.25}$$

We use notation $\gamma = \gamma^{(0)}$ for the solution of this equation. Combining Eqs. (8.25) and (8.15), we find

$$\sum_{ij} \gamma^{(0)}_{ij} \gamma^{(0)*}_{ij} = 1 \,. \tag{8.26}$$

F. From the equation for $W_{\rm bar}(\gamma)$ to the Hartree equation

Eq. (8.17) for $W_{\rm bar}(\gamma)$ is valid for any γ. If we take this equation at the saddle point $\gamma^{(0)}$ (8.25), then we arrive at the equation

$$E_0 = \sum_{mnpq} 2 V_{mnpq} \left(\sum_j \gamma^{(0)}_{mj} \gamma^{(0)*}_{nj} \right) \left(\sum_k \gamma^{(0)}_{pk} \gamma^{(0)*}_{qk} \right) \,. \tag{8.27}$$

This is nothing else but the expression for the Hartree energy (2.14) corresponding to the case $M = 2$, $L_{mn} = 0$:

$$E_0 = \frac{1}{2} \sum_{r,s=1}^{M} \sum_{n_1 n_2 n_3 n_4} V_{n_1 n_2 n_3 n_4} \phi^{r*}_{n_1} \phi^r_{n_2} \phi^{s*}_{n_3} \phi^s_{n_4} \,. \tag{8.28}$$

Indeed, taking

$$\gamma^{(0)}_{mn} = \frac{1}{\sqrt{2}} \sum_{r,s=1}^{2} \varepsilon_{rs} \left(\phi^r_m \phi^s_n\right)^* \quad (\varepsilon_{12} = -\varepsilon_{21} = 1), \tag{8.29}$$

we find

$$\sum_j \gamma^{(0)}_{mj} \gamma^{(0)*}_{nj} = \frac{1}{2} \sum_s \phi^{s*}_m \phi^s_n \,. \tag{8.30}$$

Inserting this expression into Eq. (8.27), we reproduce the Hartree expression for the energy (8.28).

G. Saddle point equation for the baryon wave function

We have explained above how to solve the Schrödinger equation for the baryon state $|B\rangle$ (8.1) described by the generating function $\tilde{\Phi}_B(\gamma)$ (8.5). Now let us turn to the vacuum. In our model the *physical* vacuum $|0\rangle$ state contains N_c quarks put into the *bare* vacuum $|\Omega\rangle$. This means that the physical vacuum of this model is described by the same equations as the baryon in simpler models with the trivial vacuum which were studied earlier. Therefore we can apply the old baryon representation (3.3) to our new vacuum:

$$|0\rangle = \sum_{i_1 \dots i_{N_c}} \psi^{i_1 \dots i_{N_c}}_{\rm vac} \prod_{c=1}^{N_c} a^+_{i_c c} |\Omega\rangle \,, \tag{8.31}$$

$$\Phi_{\rm vac}(k) = \sum_{i_1 \dots i_{N_c}} \psi^{i_1 \dots i_{N_c}}_{\rm vac} k_{i_1} \dots k_{i_{N_c}} \,. \tag{8.32}$$

Here we use notation k for the "source" argument of the function $\Phi_{\rm vac}$ instead of g as it was in Eq. (3.3).

Now we want to compute the matrix element

$$\begin{aligned}
\langle 0| \prod_c \left(\sum_j g_j a_{jc} \right) |B\rangle &= 2^{N_c} \sum_{i_m j_m} \psi^{i_1 \dots i_{N_c}}_{\rm vac} g_{j_1} \dots g_{j_{N_c}} \left(B_{\rm bar}\right)^{i_1 \dots i_{N_c}}_{j_1 \dots j_{N_c}} \\
&= \frac{1}{\left(N_c!\right)^2} \left(\sum_{ij} g_j \frac{\partial}{\partial k^*_i} \frac{\partial}{\partial \gamma_{ij}} \right)^{N_c} \left[\Phi_{\rm vac}(k)\right]^* \tilde{\Phi}_B(\gamma) \Big|_{k=\gamma=0} \\
&= \frac{N_c^{N_c}}{N_c!} \left[\exp\left(\sum_{ij} \frac{1}{N_c} g_j \frac{\partial}{\partial k^*_i} \frac{\partial}{\partial \gamma_{ij}} \right) \right] \left[\Phi_{\rm vac}(k)\right]^* \tilde{\Phi}_B(\gamma) \Big|_{k=\gamma=0} \\
= \frac{N_c^{N_c}}{N_c!} \left[\int dQ dQ^* \exp\left(-\frac{N_c}{2} \sum_{ij} Q_{ij} Q^*_{ij} \right) \right]^{-1} &\int dQ dQ^* \exp\left(-\frac{N_c}{2} \sum_{ij} Q_{ij} Q^*_{ij} \right) \\
&\times \left[\exp \sum_{ij} \left(Q^*_{ij} g_j \frac{\partial}{\partial k^*_i} + \frac{1}{2} Q_{ij} \frac{\partial}{\partial \gamma_{ij}} \right) \right] \left[\Phi_{\rm vac}(k)\right]^* \tilde{\Phi}_B(\gamma) \Big|_{k=\gamma=0} \\
&= \frac{N_c^{N_c}}{N_c!} \frac{\int dQ dQ^* \exp\left(-\frac{N_c}{2} \sum_{ij} Q_{ij} Q^*_{ij} \right) \left[\Phi_{\rm vac}(Qg^*)\right]^* \tilde{\Phi}_B(Q)}{\int dQ dQ^* \exp\left(-\frac{N_c}{2} \sum_{ij} Q_{ij} Q^*_{ij} \right)} \,.
\end{aligned} \tag{8.33}$$

Here

$$\Phi_{\rm vac}(k) = N_c^{\nu_{\rm vac}} A_{\rm vac}(k) \exp\left[N_c W_{\rm vac}(k)\right] \,, \tag{8.34}$$

$$\tilde{\Phi}_B(\gamma) = N_c^{\tilde{\nu}_B} \tilde{A}_B(\gamma) \exp\left[N_c W_{\rm bar}(\gamma)\right] \,. \tag{8.35}$$

Note that the universal functional $W_{\rm bar}(\gamma)$ describes the large N_c behavior of $\tilde{\Phi}_B(\gamma)$ for all low-lying baryons B. In the same way the functional $W_{\rm vac}(k)$ describes not only the exponential behavior of the vacuum $\Phi_{\rm vac}(k)$ but also the "meson" functionals $\Phi_{\rm mes}(\gamma)$ corresponding to the $O(N_c^0)$ excitations above the vacuum.

Thus at large N_c we have

$$\langle 0|\prod_{c=1}^{N_c}\left(\sum_j g_j a_{jc}\right)|B\rangle = \frac{e^{N_c}}{\sqrt{2\pi N_c}} N_c^{\nu_{\rm vac}+\tilde{\nu}_B} \times \frac{\int dQdQ^*\left[A_{\rm vac}(Qg^*)\right]\tilde{A}_B(Q)\exp N_c\left\{\left[W_{\rm vac}(Qg^*)\right]^* + W_{\rm bar}(Q) - \frac{1}{2}\sum\limits_{ij} Q_{ij}Q^*_{ij}\right\}}{\int dQdQ^*\exp\left(-\frac{N_c}{2}\sum\limits_{ij} Q_{ij}Q^*_{ij}\right)}\,. \tag{8.36}$$

At large N_c the integral over Q, Q^* can be computed using the saddle point method. The deformation of the integration contour in the saddle point method can lead to the violation of the complex conjugation relation between Q_{ij} and Q^*_{ij}. Therefore we introduce an independent notation for Q^*_{ij}

$$Q^*_{ij} \to u_{ij}\,. \tag{8.37}$$

The saddle point equations are

$$\frac{\partial}{\partial u_{ij}}\left\{\left[W_{\rm vac}(u^*g^*)\right]^* + W_{\rm bar}(Q) - \frac{1}{2}\sum_{mn} Q_{mn}u_{mn}\right\} = 0\,, \tag{8.38}$$

$$\frac{\partial}{\partial Q_{ij}}\left\{\left[W_{\rm vac}(u^*g^*)\right]^* + W_{\rm bar}(Q) - \frac{1}{2}\sum_{mn} Q_{mn}u_{mn}\right\} = 0\,. \tag{8.39}$$

Now we obtain from Eqs. (8.38), (8.39)

$$k_i = u^*_{ij}g^*_j\,, \tag{8.40}$$

$$g_j\left[\frac{\partial W_{\rm vac}(k)}{\partial k_i}\right]^* - g_i\left[\frac{\partial W_{\rm vac}(k)}{\partial k_j}\right]^* = Q_{ij}\,, \tag{8.41}$$

$$\frac{\partial W_{\rm bar}(Q)}{\partial Q_{ij}} = u_{ij}\,. \tag{8.42}$$

These equations determine

$$Q_{ij} = Q_{ij}(g)\,, \quad u_{ij} = u_{ij}(g)\,. \tag{8.43}$$

Combining Eqs. (8.42) and (8.15), we find

$$\sum_{ij} Q_{ij}(g)u_{ij}(g) = 2\,. \tag{8.44}$$

H. Functionals $W(g)$ and $A_B(g)$ from the saddle point method

Applying the saddle-point method to Eq. (8.36), we find

$$\langle 0|\prod_{c=1}^{N_c}\left(\sum_j g_j a_{jc}\right)|B\rangle = N_c^{\nu_B} J(g)\left[A_{\rm vac}(u^*g^*)\right]^*\tilde{A}_B(Q) \times \exp N_c\left\{\left[W_{\rm vac}(u^*g^*)\right]^* + W_{\rm bar}(Q_{ij})\right\}_{Q=Q(g),u=u(g)}\,. \tag{8.45}$$

Here $J(g)$ stands from the contribution of the Jacobian corresponding to the fluctuations around the saddle point and $N_c^{\nu_B}$ accumulates all powers of N_c coming from various sources. We can rewrite the above representation in the form

$$\langle 0|\prod_{c=1}^{N_c}\left(\sum_j g_j a_{jc}\right)|B\rangle = N_c^{\nu_B} A_B(g)\exp\left[N_c W(g)\right]\,, \tag{8.46}$$

where

$$W(g) = \left\{ [W_{\rm vac}(u^* g^*)]^* + W_{\rm bar}(Q_{ij}) \right\}_{Q=Q(g),u=u(g)} , \tag{8.47}$$

$$A_B(g) = J(g) \left[A_{\rm vac}(u^* g^*) \right]^* \tilde{A}_B(Q) \Big|_{Q=Q(g),u=u(g)} . \tag{8.48}$$

Using the factorization of $\tilde{A}_B(Q_{ij})$ (8.20), we arrive at the factorized form of $A_B(g)$

$$A_B(g) = A^{(0)}(g) \prod_k A_k(g) \tag{8.49}$$

with

$$A_k(g) = \xi_k \left(Q(g) \right) , \tag{8.50}$$

$$A^{(0)}(g) = J(g) \tilde{A}^{(0)} \left(Q(g) \right) \left\{ A_{\rm vac} \left([u(g) g]^* \right) \right\}^* . \tag{8.51}$$

Our result (8.49) shows the mechanism of the general factorization (1.18) of functionals $A_B(g)$ in models with the nontrivial vacuum.

I. Representations for $W(g)$ in terms of trajectories

In our current model, the physical vacuum state contains N_c quarks occupying the bare vacuum. This means our physical *vacuum* can be described by equations derived in Sec. VI for *baryons*. Thus the functional $W_{\rm vac}$ is given by Eqs. (6.39) and (6.42). In the special case $L_{mn} = 0$ with which we are dealing now [See Eq. (8.11)], the result for $W_{\rm vac}$ is especially simple and is given by Eq. (6.57):

$$W_{\rm vac}(k) = \ln I_{\rm vac}(k) \quad (\text{if } L_{mn} = 0) , \tag{8.52}$$

where $I_{\rm vac}(g)$ is determined by the trajectories obeying asymptotic conditions (6.58), (6.59) and (6.60)

$$q_n^{\rm vac}(t) \overset{t \to -\infty}{=} \delta_{n0} I_{\rm vac}(k) \exp \left(\varepsilon_{\rm vac}^1 t \right) , \tag{8.53}$$

$$p_n^{\rm vac}(t) \overset{t \to -\infty}{=} \frac{\delta_{n0}}{I_{\rm vac}(k)} \exp \left(-\varepsilon_{\rm vac}^1 t \right) , \tag{8.54}$$

$$q_n^{\rm vac}(0) = k_n . \tag{8.55}$$

These trajectories obey condition (6.25):

$$\sum_n p_n^{\rm vac}(t) q_n^{\rm vac}(t) = 1 . \tag{8.56}$$

According to Eqs. (6.30), (6.41) and (6.42) we have

$$p_n^{\rm vac}(0) = \frac{\partial W_{\rm vac}(k)}{\partial k_n} . \tag{8.57}$$

One can generalize the work done in Sec. VI for the case of states containing N_c quarks above the bare vacuum and derive a similar representation for $W_{\rm bar}(g)$:

$$W_{\rm bar}(Q) = \ln I_{\rm bar}(Q) \quad (\text{if } L_{mn} = 0) . \tag{8.58}$$

Here Q_{ij} is the boundary $t = 0$ value of some trajectory $Q_{ij}^{\rm bar}(t)$ in the space of antisymmetric tensors

$$Q_{ij}^{\rm bar}(0) = Q_{ij} . \tag{8.59}$$

and the parameter $I_{\rm bar}(Q)$ is determined by the $t \to -\infty$ asymptotic behavior of these trajectories

$$Q_{ij}^{\rm bar}(t) \overset{t \to -\infty}{=} \varepsilon_{ij} I_{\rm bar}(Q) \exp \left[\left(\varepsilon_{\rm bar}^1 + \varepsilon_{\rm bar}^2 \right) t \right] , \tag{8.60}$$

$$P_{ij}^{\rm bar}(t) \overset{t \to -\infty}{=} \frac{\varepsilon_{ij}}{2 I_{\rm bar}(Q)} \exp \left[- \left(\varepsilon_{\rm bar}^1 + \varepsilon_{\rm bar}^2 \right) t \right] . \tag{8.61}$$

Here ε_{ij} is the antisymmetric tensor

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } i=1, j=2, \\ -1 & \text{if } i=2, j=1, \\ 0 & \text{otherwise}. \end{cases} \tag{8.62}$$

We assume that the values $i = 1, 2$ correspond to the two occupied single-particle levels of the baryon state in the Hartree picture. The Hamilton equation describing the dynamics in the space of antisymmetric tensors $Q^{\rm bar}_{ij}$, $P^{\rm bar}_{ij}$ will be discussed in Sec. X. By analogy with Eq. (8.57) we have

$$P^{\rm bar}_{ij}(0) = \frac{1}{2}\frac{\partial W_{\rm bar}(Q)}{\partial Q_{ij}}\,. \tag{8.63}$$

The analog of Eq. (8.56) is

$$\sum_{ij} P^{\rm bar}_{ij}(t) Q^{\rm bar}_{ij}(t) = 1\,. \tag{8.64}$$

The constant on the RHS of Eq. (8.63) is fixed by relations (8.15).

The above representations (8.52) for $W_{\rm vac}(k)$ and (8.58) for $W_{\rm bar}(Q)$ are independent of each other. However, in the expression (8.47) for the functional $W(g)$

$$W_{\rm Sch}(g) = \{W_{\rm vac}\,[k(g^*)]\}^* + W_{\rm bar}\,[Q(g)] \tag{8.65}$$

the variables Q and k on the RHS become g (g^*) dependent. We use the label Sch in $W_{\rm Sch}(g)$ in order to stress that we are dealing with the Schrödinger approach. Later we will show how the functional $W(g)$ can be calculated in the path integral approach [using notation $W_{\rm pi}(g)$]. Certainly the result of the calculation should not depend on the method but in order to avoid confusion at the intermediate steps we prefer to use different notations for the two different methods.

The dependence of k and Q on g is described by equations (8.40)–(8.42):

$$\sum_j g_j \frac{\partial W_{\rm bar}(Q)}{\partial Q_{ij}} = k^*_i\,, \tag{8.66}$$

$$g_j \left[\frac{\partial W_{\rm vac}(k)}{\partial k_i}\right]^* - g_i \left[\frac{\partial W_{\rm vac}(k)}{\partial k_j}\right]^* = Q_{ij}\,. \tag{8.67}$$

Thus the problem of the calculation of the functional $W_{\rm Sch}(g)$ reduces (in the case $L_{mn} = 0$) to search for trajectories obeying boundary conditions (8.53)–(8.55), (8.59)–(8.61), (8.66), (8.67). The asymptotic behavior of these trajectories at $t \to \infty$ determines parameters $I_{\rm vac}(k)$ and $I_{\rm bar}(Q)$. According to Eqs. (8.52), (8.58) and (8.65) we have

$$W_{\rm Sch}(g) = \{\ln I_{\rm vac}\,[k(g^*)]\}^* + \ln I_{\rm bar}\,[Q(g)] \quad (\text{if } L_{mn} = 0)\,. \tag{8.68}$$

J. Generalization for models with $M > 1$

The previous analysis was devoted to "baryon" states containing $2N_c$ quarks above the bare vacuum. These states were described by functions $\tilde{\Phi}_B(\gamma)$ depending on antisymmetric tensors γ_{ij}. The generalization to the case of states with MN_c quarks is straightforward. In this case we must work with wave functions depending on antisymmetric tensors $\gamma_{[i_1 i_2 \ldots i_M]}$ of rank k. The analog of Eq. (8.8) is

$$T_{mn} = \sum_{c=1}^{N_c} a^+_{mc} a_{nc} \to \frac{1}{(M-1)!} \sum_{i_2 i_3 \ldots i_M} \gamma_{m i_2 i_3 \ldots i_M} \frac{\partial}{\partial \gamma_{n i_2 i_3 \ldots i_M}}\,, \tag{8.69}$$

where the derivative is normalized by the condition

$$\frac{\partial}{\partial \gamma_{i_1 i_2 i_3 \ldots i_M}} \gamma_{j_1 j_2 j_3 \ldots j_M} = \det_{ab} \|\delta_{i_a j_b}\|\,. \tag{8.70}$$

IX. PATH INTEGRAL APPROACH

A. Advantage of the path integral approach

Our previous analysis of the large-N_c models was based on the operator approach with the stationary Schrödinger equation (2.2) as a starting point. Now we want to study large-N_c models using the path integral method. In principle, both Schrödinger and path integral approaches must lead to the same results. In Sec. X we will demonstrate the equivalence of the results derived using these two methods. However, in spite of this equivalence we will see that the path integral method allows us to obtain in a straightforward way rather interesting results which are not so obvious in the Schrödinger approach. The reason is that the path integral approach is based on time-dependent trajectories. On the other hand, as we know from Sec. VI, in the Schrödinger approach the trajectories appear rather indirectly. In the Schrödinger approach one first derives the Hamilton–Jacobi equation for the "action" $W(g)$ and only after that $W(g)$ can be interpreted in terms of trajectories. In the case of models with the nontrivial vacuum the situation is even more involved, since the trajectories appearing in the Schrödinger approach belong to the phase space of antisymmetric tensors. On the contrary, the classical dynamics coming from the path integral approach can be formulated in simpler terms.

Since the path integral approach allows for a straightforward derivation of the representation for the functional $W(g)$ in terms of simple classical dynamics, we prefer to turn to the path integral formalism in this section. The equivalence of the results obtained in this section with the results based on the Schrödinger equation will be established in Sec. X.

B. Model

We want to apply the path integral approach to models described by the Hamiltonian (2.8). For simplicity we will consider the case when $L_{n_1n_2} = 0$ in this Hamiltonian. For our aims it is convenient to rewrite the Hamiltonian in the form

$$H = \frac{1}{2N_c} \sum_{\alpha\beta} \left(a^+\Gamma_\alpha a\right) V_{\alpha\beta} \left(a^+\Gamma_\beta a\right) \,, \tag{9.1}$$

where

$$\left(a^+\Gamma_\alpha a\right) = \sum_{c=1}^{N_c} \sum_{mn} a^+_{cm}\Gamma^{mn}_\alpha a_{cn} \,. \tag{9.2}$$

The coefficients $V_{\alpha\beta}$ and Γ^{mn}_α are assumed to obey the conditions

$$V_{\alpha\beta} = V^*_{\alpha\beta} \,, \quad V_{\alpha\beta} = V_{\beta\alpha} \,, \tag{9.3}$$

$$(\Gamma^{mn}_\alpha)^* = \Gamma^{nm}_\alpha \,. \tag{9.4}$$

The expression (9.1) for the Hamiltonian corresponds to the choice

$$V_{n_1n_2n_3n_4} = \sum_\alpha \Gamma^{n_1n_2}_\alpha \Gamma^{n_3n_4}_\alpha \tag{9.5}$$

in Eq. (2.8) .

In the path integral approach this system is described by the action

$$S_E(a) = -\int dt \left[\left(a^+\partial_t a\right) + \frac{1}{2N_c} \sum_{\alpha\beta} \left(a^+\Gamma_\alpha a\right) V_{\alpha\beta} \left(a^+\Gamma_\beta a\right) \right] . \tag{9.6}$$

We use the same notation a^+, a for the Grassmann integration variables as for the operators in Eq. (9.1). Strictly speaking, the transition from the operator formalism to the path integral has to be accompanied by a careful treatment of the problems of the operator ordering. However, in the leading order of the $1/N_c$ expansion these subtleties are not important.

One can "bosonize" the theory introducing the path integral over the auxiliary boson variable $\pi(t)$:

$$\exp S_E(a) = \frac{\int D\pi \exp\left\{-\left[a^+ K(\pi) a\right] + N_c S_{\rm bos}(\pi)\right\}}{\int D\pi \exp\left[N_c S_{\rm bos}(\pi)\right]} \,. \tag{9.7}$$

Here

$$S_{\rm bos}(\pi) = \int dt L_{\rm bos}(\pi) \,, \tag{9.8}$$

$$L_{\rm bos}(\pi) = \sum_{\alpha\beta} \frac{1}{2} \pi_\alpha \left(V^{-1}\right)_{\alpha\beta} \pi_\beta \,, \tag{9.9}$$

$$\left[a^+ K(\pi) a\right] = \int dt a^+ K(\pi) a \,, \tag{9.10}$$

$$K(\pi) = \partial_t + \sum_\alpha \pi_\alpha \Gamma_\alpha \,. \tag{9.11}$$

C. Path integral approach at finite N_c

The generating functional $\Phi_B(g)$ for the baryon wave function (2.6) can be represented in the form

$$\Phi_B(g) = \langle 0 | J(g, a, 0) | B \rangle \,, \tag{9.12}$$

where we use the short notation

$$J(g, a, t_1) = \prod_{c=1}^{N_c} \left[\sum_m g_m \cdot a_{mc}(t_1) \right] \,, \tag{9.13}$$

$$J(g', a^+, t_2) = \prod_{c=1}^{N_c} \left[\sum_m g'_m \cdot a^+_{mc}(t_2) \right] \,. \tag{9.14}$$

We work with the Euclidean time. Therefore operators $a(t)$ and $a^+(t)$ are not Hermitean conjugate.

In order to study $\Phi_B(g)$ in the path integral approach we introduce the correlation function

$$Z(g, g', t_1, t_2) = \langle 0 | T \left\{ J(g, a, t_1) J(g', a^+, t_2) \right\} | 0 \rangle \,. \tag{9.15}$$

The path integral representation for this correlation function is

$$Z(g, g', t_1, t_2) = \frac{\int DaDa^+ D\pi J(g, a, t_1) J(g', a^+, t_2) \exp\left\{-\left[a^+ K(\pi) a\right] + N_c S_{\rm bos}(\pi)\right\}}{\int DaDa^+ D\pi \exp\left\{-\left[a^+ K(\pi) a\right] + N_c S_{\rm bos}(\pi)\right\}} \,. \tag{9.16}$$

At large $t_1 - t_2 \to +\infty$ only the contribution of the lightest baryon $|B\rangle$ survives in Eq. (9.15):

$$Z(g, g', t_1, t_2) \stackrel{t_1 - t_2 \to +\infty}{\longrightarrow} \langle 0 | J(g, a, 0) | B \rangle e^{-(t_2 - t_1)(\mathcal{E}_B - \mathcal{E}_{\rm vac})} \langle B | J(g', a^+, 0) | 0 \rangle \,. \tag{9.17}$$

D. Large N_c limit

Now we want to consider the limit of large N_c. In this case we have with the exponential accuracy

$$\langle 0 | J(g, a, 0) | B \rangle \sim e^{N_c W(g)} \,. \tag{9.18}$$

The difference $\mathcal{E}_B - \mathcal{E}_{\rm vac}$ appearing in Eq. (9.17) has the order $O(N_c)$:

$$\mathcal{E}_B - \mathcal{E}_{\rm vac} = N_c \left(E_{\rm bar} - E_{\rm vac}\right) + O(N_c^0) \,. \tag{9.19}$$

If we combine the large-time limit with the large-N_c limit

$$t_1 - t_2 \to \infty, \quad N_c \to \infty \,, \tag{9.20}$$

then we find from Eqs. (9.17), (9.18) and (9.19)

$$Z(g, g', t_1, t_2) \sim \exp\left\{ N_c \left[-\left(E_{\rm bar} - E_{\rm vac}\right)(t_1 - t_2) + W(g) + \left[W(g'^*)\right]^* \right] \right\} \,. \tag{9.21}$$

E. Calculation of the path integral

First we can compute the Gaussian integral over quarks in Eq. (9.16)

$$Z(g,g',t_1,t_2)=\frac{\int D\pi\left\{\left[g\cdot\langle t_1|K^{-1}(\pi)|t_2\rangle\cdot g'\right]\left[\mathrm{Det}K(\pi)\right]\exp\left[S_{\rm bos}(\pi)\right]\right\}^{N_c}}{\int D\pi\left\{\left[\mathrm{Det}K(\pi)\right]\exp S_{\rm bos}(\pi)\right\}^{N_c}}\,. \tag{9.22}$$

At large N_c the path integrals in the numerator and in the denominator can be computed using the saddle point method. The numerator and the denominator have different saddle points which will be denoted $\pi^{\rm cl}$ and $\pi^{\rm vac}$, respectively. The result of the saddle-point integration is

$$\frac{1}{N_c}\ln Z(g,g',t_1,t_2)=S_{\rm nonloc}(\pi^{\rm cl})+\Delta S_{\rm bos}\left(\pi^{\rm cl}\right)\,, \tag{9.23}$$

where

$$S_{\rm nonloc}(\pi)=\ln\left\{\left[g\cdot\langle t_1|K^{-1}(\pi)|t_2\rangle\cdot g'\right]\mathrm{Det}\frac{K(\pi)}{K(\pi^{\rm vac})}\right\}\,, \tag{9.24}$$

$$\Delta S_{\rm bos}\left(\pi\right)=S_{\rm bos}(\pi)-S_{\rm bos}(\pi^{\rm vac})=\int_{-\infty}^{\infty}dt\left[L_{\rm bos}(\pi)-L_{\rm bos}(\pi^{\rm vac})\right]\,. \tag{9.25}$$

The saddle point $\pi^{\rm cl}$ is given by the equation

$$\frac{\delta}{\delta\pi}\left[S_{\rm nonloc}(\pi)+\Delta S_{\rm bos}\left(\pi\right)\right]\Big|_{\pi=\pi^{\rm cl}}=0\,. \tag{9.26}$$

The solution $\pi^{\rm cl}$ is t dependent. It also depends on g,g',t_1,t_2:

$$\pi^{\rm cl}=\pi^{\rm cl}(t|g,g',t_1,t_2)\,. \tag{9.27}$$

On the contrary, the saddle point $\pi^{\rm vac}$ is a constant which can be found from the saddle point equation

$$\frac{\delta}{\delta\pi}\ln\left\{\left[\mathrm{Det}K(\pi)\right]\left[\exp S_{\rm bos}(\pi)\right]\right\}\Big|_{\pi=\pi^{\rm vac}}=0\,. \tag{9.28}$$

At large times $t\gg t_1$ or $t\ll t_2$ we have

$$\lim_{t\to\pm\infty}\pi^{\rm cl}(t)=\pi^{\rm vac}\,. \tag{9.29}$$

Combining Eqs. (9.21) and (9.23), we find

$$W(g)+\left[W(g'^*)\right]^*=\lim_{t_1-t_2\to+\infty}\left[S_{\rm nonloc}(\pi^{\rm cl})+\Delta S_{\rm bos}\left(\pi^{\rm cl}\right)+(t_1-t_2)\left(E_{\rm bar}-E_{\rm vac}\right)\right]\,. \tag{9.30}$$

F. Fermion formulation of the effective large-N_c theory

In principle, Eq. (9.30) combined with the saddle point equation (9.26) contains everything what is needed for the calculation of the functional $W(g)$. However, these equations involve the nonlocal action $S_{\rm nonloc}(\pi^{\rm cl})$ defined in Eq. (9.24). In order to solve the saddle point equation (9.26) we would like to find a local formulation for our effective large-N_c theory. To this aim we write the path integral representation for $S_{\rm nonloc}(\pi)$ over auxiliary Grassmann variables b, b^+:

$$\begin{aligned}\exp S_{\rm nonloc}(\pi)&=\left[g\cdot\langle t_1|K^{-1}(\pi^{\rm cl})|t_2\rangle\cdot g'\right]\mathrm{Det}\frac{K(\pi)}{K(\pi^{\rm vac})}\\&=\frac{\int DbDb^+\left[gb(t_1)\right]\left[g'b^+(t_2)\right]\exp\tilde S(b,b^+,\pi)}{\int DbDb^+\exp\tilde S(b,b^+,\pi^{\rm vac})}\,.\end{aligned} \tag{9.31}$$

Note that the fermion fields b, b^+ appearing here have *no color* indices. The action $\tilde{S}$ is

$$\tilde{S}(b,b^+,\pi)=\int dt\left[b^+K(\pi)b\right]\,. \tag{9.32}$$

The path integral (9.31) can be also interpreted in terms of the operator approach to the theory of quantum fermions in the background classical field π

$$\frac{\int DbDb^+\left[gb(t_1)\right]\left[g'b^+(t_2)\right]\exp\tilde{S}(b,b^+,\pi)}{\int DbDb^+\exp\tilde{S}(b,b^+,\pi^{\rm vac})}=\langle 0|\left[gb(t_1)\right]\left[g'b^+(t_2)\right]|0\rangle_\pi\,. \tag{9.33}$$

For brevity we omit the *in* and *out* labels of the two "vacuum" states in this matrix element. One should also keep in mind that the Euclidean evolution violates the Hermitian conjugation of operator $b(t)$ and $b^+(t)$. Moreover even the hermiticity of the Hamiltonian can be violated. Indeed, the background field $\pi^{(1)}$ comes from the saddle point equation (9.26) and the application of the saddle point method can be accompanied by the deformation of the integration contour.

Combining Eqs. (9.31) and (9.33), we find

$$S_{\rm nonloc}(\pi)=\ln\langle 0|\left[gb(t_1)\right]\left[g'b^+(t_2)\right]|0\rangle_\pi\,. \tag{9.34}$$

Inserting Eq. (9.34) into Eq. (9.30), we obtain

$$W(g)+\left[W(g'^*)\right]^*=\lim_{t_1-t_2\to+\infty}\left\{\ln\langle 0|\left[gb(t_1)\right]\left[g'b^+(t_2)\right]|0\rangle_{\pi^{\rm cl}}+\Delta S_{\rm bos}\left(\pi^{\rm cl}\right)-(t_1-t_2)\left(E_{\rm bar}-E_{\rm vac}\right)\right\}\,. \tag{9.35}$$

Using Eqs. (9.8), (9.9), and (9.34), we find from the saddle point equation (9.26)

$$\left\{\frac{\delta}{\delta\pi_\alpha(t)}\langle 0|\left[gb(t_1)\right]\left[g'b^+(t_2)\right]|0\rangle_\pi\right\}_{\pi=\pi^{\rm cl}}=-\sum_\beta\left(V^{-1}\right)_{\alpha\beta}\left(\pi^{\rm cl}\right)_\beta(t)\langle 0|\left[gb(t_1)\right]\left[g'b^+(t_2)\right]|0\rangle_{\pi^{\rm cl}}\,. \tag{9.36}$$

The variational derivative on the LHS has a simple interpretation in terms of matrix elements in the operator formulation of the same theory:

$$\frac{\delta}{\delta\pi_\alpha(t)}\langle 0|\left[gb(t_1)\right]\left[g'b^+(t_2)\right]|0\rangle_\pi=-\langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\alpha b(t)\right]\left[g'b^+(t_2)\right]\right\}|0\rangle_\pi\,. \tag{9.37}$$

Combining equations (9.36) and (9.37), we find

$$\begin{aligned}&\sum_\beta\left(V^{-1}\right)_{\alpha\beta}\left(\pi^{\rm cl}\right)_\beta(t)\langle 0|\left[gb(t_1)\right]\left[g'b^+(t_2)\right]|0\rangle_{\pi^{\rm cl}}\\&=\langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\alpha b(t)\right]\left[g'b^+(t_2)\right]\right\}|0\rangle_{\pi^{\rm cl}}\,.\end{aligned} \tag{9.38}$$

This is a new form of the saddle point equation which determines $\pi^{\rm cl}$.

G. Separation of the t_1 and t_2 contributions

Our aim is to compute the functional $W(g)$. However, representation (9.35) is written for the sum $W(g)+[W(g'^*)]^*$. Now we want to extract information about $W(g)$ from this equation. At finite t_1, t_2 the RHS of Eq. (9.35) is a nontrivial functional of g and g'. Only in the limit $t_1-t_2\to+\infty$ the dependence on g and g' reduces to the sum of two independent terms $W(g)+[W(g'^*)]^*$. This additive dependence is a consequence of the properties of the classical solution $\pi^{\rm cl}(t)$ at large t_1 and t_2.

Eq. (9.35) was derived in the limit $t_1-t_2\to+\infty$. But if we look at the history of the derivation of this equation, then we find that the limit $t_1-t_2\to+\infty$ was used only on the RHS of this equation. In particular, the saddle point equation (9.38) does not know anything about the limit $t_1-t_2\to+\infty$.

According to Eq. (9.29) at large $t\to\pm\infty$ (when $t\gg t_1$ or $t\ll t_2$) the field $\pi^{\rm cl}(t)$ approaches the constant configuration $\pi^{\rm vac}$. At large $t_1-t_2\to+\infty$ we can also consider the intermediate region $t_2\ll t\ll t_1$. In this region the field $\pi^{\rm cl}(t)$ is also asymptotically static but with another value $\pi^{\rm bar}$. Thus we have three regions where $\pi^{\rm cl}(t)$ becomes asymptotically static:

$$\pi^{\rm cl}(t)=\begin{cases}\pi^{\rm vac}, & \text{if } t\ll t_2,\\ \pi^{\rm bar} & \text{if } t_2\ll t\ll t_1,\\ \pi^{\rm vac} & \text{if } t_1\ll t\,.\end{cases} \tag{9.39}$$

The constant configuration $\pi^{\rm bar}$ corresponds to the "baryon" state $|B\rangle$ of the effective fermion theory (remember that now we deal with the effective fermion theory where fermions have no color indices). Therefore the matrix elements appearing in Eq. (9.38) become

$$\langle 0|T\left\{\left[gb(t_1)\right]\left[g'b^+(t_2)\right]\right\}|0\rangle_{\pi^{\rm cl}} \overset{t_1\gtrless t_2}{=} \langle 0|\left[gb(t_1)\right]|B\rangle_\pi\langle B|\left[g'b^+(t_2)\right]|0\rangle_{\pi^{\rm cl}}\ , \tag{9.40}$$

$$\begin{aligned}&\langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\alpha b(t)\right]\left[g'b^+(t_2)\right]\right\}|0\rangle_{\pi^{\rm cl}}\\ &\overset{t_1\sim t\gtrless t_2}{=} \langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\alpha b(t)\right]\right\}|B\rangle_{\pi^{\rm cl}}\langle B|\left[g'b^+(t_2)\right]|0\rangle_{\pi^{\rm cl}}\,.\end{aligned} \tag{9.41}$$

Inserting these factorized expressions into the saddle point equation (9.38), we find in the region $t_1\sim t\gg t_2$

$$\sum_\beta \left(V^{-1}\right)_{\alpha\beta}\pi_\beta(t)\langle 0|\left[gb(t_1)\right]|B\rangle_{\pi^{\rm cl}} \overset{t_1\sim t\gg t_2}{=} \langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\alpha b(t)\right]\right\}|B\rangle_{\pi^{\rm cl}}\,. \tag{9.42}$$

Let us take the limit $t_2\to-\infty$ keeping t_1 and t fixed. In this limit Eq. (9.42) "forgets" about the behavior of $\pi(t)$ at $t\sim t_2$ and at $t\ll t_2$. Hence we can replace $\pi^{\rm cl}$ by its t_1 component

$$\pi^{(1)}(t)=\begin{cases}\pi^{\rm bar} & \text{if } t\ll t_1,\\ \pi^{\rm cl}(t) & \text{if } t\sim t_1,\\ \pi^{\rm vac} & \text{if } t\gg t_1.\end{cases} \tag{9.43}$$

As a result, Eq. (9.42) becomes

$$\sum_\beta \left(V^{-1}\right)_{\alpha\beta}\pi^{(1)}_\beta(t)\langle 0|\left[gb(t_1)\right]|B\rangle_{\pi^{(1)}} = \langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\alpha b(t)\right]\right\}|B\rangle_{\pi^{(1)}}\,. \tag{9.44}$$

Hence

$$\pi^{(1)}_\alpha(t)=\sum_\beta V_{\alpha\beta}\frac{\langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\beta b(t)\right]\right\}|B\rangle_{\pi^{(1)}}}{\langle 0|\left[gb(t_1)\right]|B\rangle_{\pi^{(1)}}}\,. \tag{9.45}$$

Working with the background field $\pi^{(1)}$, we can consider the limits $t\to\pm\infty$ at fixed t_1

$$\langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\alpha b(t)\right]\right\}|B\rangle_{\pi^{(1)}} \overset{t\lessgtr t_1}{=} \langle 0|\left[gb(t_1)\right]|B\rangle_{\pi^{(1)}}\langle B|\left[b^+(t)\Gamma_\alpha b(t)\right]|B\rangle_{\pi^{\rm bar}}\,, \tag{9.46}$$

$$\langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\alpha b(t)\right]\right\}|B\rangle_{\pi^{(1)}} \overset{t\gtrless t_1}{=} \langle 0|\left[b^+(t)\Gamma_\alpha b(t)\right]|0\rangle_{\pi^{\rm vac}}\langle 0|\left[gb(t_1)\right]|B\rangle_{\pi^{(1)}}\,. \tag{9.47}$$

Inserting these asymptotic expressions into Eq. (9.45), we obtain

$$\pi^{\rm vac}_\alpha=\sum_\beta V_{\alpha\beta}\langle 0|\left[b^+(0)\Gamma_\beta b(0)\right]|0\rangle_{\pi^{\rm vac}}\,, \tag{9.48}$$

$$\pi^{\rm bar}_\alpha=\sum_\beta V_{\alpha\beta}\langle B|\left[b^+(0)\Gamma_\beta b(0)\right]|B\rangle_{\pi^{\rm bar}}\,. \tag{9.49}$$

In Sec. IX H we will show that these are nothing else but the static Hartree equations for the vacuum and for the baryon in the large-N_c theory.

Note that the time-ordered product has a discontinuity at $t=t_1$:

$$\left[\lim_{t\to t_1-0}-\lim_{t\to t_1+0}\right]\langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\alpha b(t)\right]\right\}|B\rangle_{\pi^{(1)}} = \langle 0|\left[g\Gamma_\alpha b(t_1)\right]|B\rangle_{\pi^{(1)}}\,. \tag{9.50}$$

Combining this with Eq. (9.45), we conclude that the field $\pi^{(1)}(t)$ has a discontinuity at $t\to t_1$:

$$\left[\lim_{t\to t_1-0}-\lim_{t\to t_1+0}\right]\pi^{(1)}_\alpha(t)=\sum_\beta V_{\alpha\beta}\frac{\langle 0|\left[g\Gamma_\beta b(t_1)\right]|B\rangle_{\pi^{(1)}}}{\langle 0|\left[gb(t_1)\right]|B\rangle_{\pi^{(1)}}}\,. \tag{9.51}$$

Using the property (9.40), we can rewrite Eq. (9.35) in the form

$$W(g) + [W(g'^*)]^* = \ln\langle 0|\,[gb(t_1)]\,|B\rangle_{\pi^{(1)}} + \ln\langle B|\,\left[g'b^+(t_2)\right]\,|0\rangle_{\pi^{(2)}} + \Delta S_{\rm bos}\left(\pi^{\rm cl}\right) - (t_1-t_2)\,(E_{\rm bar}-E_{\rm vac})\,. \quad (9.52)$$

Here $\pi^{(2)}$ is the analog of $\pi^{(1)}$ for the vicinity of the t_2 time epoch:

$$\pi^{(2)}(t) = \begin{cases} \pi^{\rm vac} & \text{if } t \ll t_2, \\ \pi^{\rm cl}(t) & \text{if } t \sim t_2, \\ \pi^{\rm bar} & \text{if } t \gg t_2. \end{cases} \quad (9.53)$$

Using the definition (9.25) of $\Delta S_{\rm bos}$ and the properties (9.43), (9.53) of $\pi^{(1)}$ and $\pi^{(2)}$, we obtain

$$\begin{aligned} \Delta S_{\rm bos}\left(\pi^{\rm cl}\right) &= \int_{-\infty}^{t_2} dt\,\left[L_{\rm bos}(\pi^{\rm cl}) - L_{\rm bos}(\pi^{\rm vac})\right] + \int_{t_2}^{t_1} dt\,\left[L_{\rm bos}(\pi^{\rm cl}) - L_{\rm bos}(\pi^{\rm bar})\right] \\ &+ \int_{t_1}^{\infty} dt\,\left[L_{\rm bos}(\pi^{\rm cl}) - L_{\rm bos}(\pi^{\rm vac})\right] + \left[L_{\rm bos}(\pi^{\rm bar}) - L_{\rm bos}(\pi^{\rm vac})\right](t_1-t_2) \\ &= \int_{-\infty}^{t_2} dt\,\left[L_{\rm bos}(\pi^{(2)}) - L_{\rm bos}(\pi^{\rm vac})\right] + \int_{t_2}^{+\infty} dt\,\left[L_{\rm bos}(\pi^{(2)}) - L_{\rm bos}(\pi^{\rm bar})\right] \\ &+ \int_{-\infty}^{t_1} dt\,\left[L_{\rm bos}(\pi^{(1)}) - L_{\rm bos}(\pi^{\rm bar})\right] + \int_{t_1}^{\infty} dt\,\left[L_{\rm bos}(\pi^{(1)}) - L_{\rm bos}(\pi^{\rm vac})\right] \\ &+ \left[L_{\rm bos}(\pi^{\rm bar}) - L_{\rm bos}(\pi^{\rm vac})\right](t_1-t_2)\,. \end{aligned} \quad (9.54)$$

Inserting this into Eq. (9.52) and separating the t_1 and t_2 contributions, we find

$$\begin{aligned} W(g) &= \ln\langle 0|\,[gb(t_1)]\,|B\rangle_{\pi^{(1)}} + \int_{-\infty}^{t_1} dt\,\left[L_{\rm bos}(\pi^{(1)}) - L_{\rm bos}(\pi^{\rm bar})\right] + \int_{t_1}^{\infty} dt\,\left[L_{\rm bos}(\pi^{(1)}) - L_{\rm bos}(\pi^{\rm vac})\right] \\ &+ t_1\left\{(E_{\rm bar}-E_{\rm vac}) + \left[L_{\rm bos}(\pi^{\rm bar}) - L_{\rm bos}(\pi^{\rm vac})\right]\right\}\,. \end{aligned} \quad (9.55)$$

Thus we have reduced the problem of the calculation of the functional $W(g)$ to the analysis of the fermion system in the "self-consistent" field $\pi^{(1)}(t)$. Indeed, the field $\pi^{(1)}(t)$ is determined by Eq. (9.45) combined with the discontinuity condition (9.51) at t_1 and with the boundary conditions (9.43)

$$\lim_{t\to-\infty}\pi^{(1)}(t) = \pi^{\rm bar}\,, \quad \lim_{t\to+\infty}\pi^{(1)}(t) = \pi^{\rm vac}\,. \quad (9.56)$$

Inserting this field $\pi^{(1)}(t)$ into Eq. (9.55), we can find $W(g)$. Note that the g dependence appears via the discontinuity (9.51) of $\pi^{(1)}(t)$ at t_1.

H. Hartree equations from the path integral formalism

The saddle point field $\pi^{(1)}(t)$ is determined by equation (9.45). This equation contains the matrix elements

$$\langle 0|\,[gb(t_1)]\,|B\rangle_{\pi^{(1)}}\,, \quad (9.57)$$

$$\langle 0|T\left\{[gb(t_1)]\,\left[b^+(t)\Gamma_\beta b(t)\right]\right\}|B\rangle_{\pi^{(1)}}\,. \quad (9.58)$$

of the effective theory with the b fermions that have no color.

The matrix elements (9.57) and (9.58) are written for the theory in the background t-dependent field $\pi^{(1)}$. The vacuum state $\langle 0|$ appears in these matrix elements as an *out* state corresponding to $t\to+\infty$. At $t\to+\infty$ we have $\pi^{(1)}(t)\to\pi^{\rm vac}$ according to Eq. (9.56). Therefore the state $\langle 0|$ is the ground state of the theory with the Hamiltonian associated with the quadratic form (9.11) taken in the static background field $\pi^{(1)}$:

$$\langle 0|\sum_\alpha \pi^{\rm vac}_\alpha\left(b^+\Gamma_\alpha b\right) = \langle 0|E_{\rm vac}\,. \quad (9.59)$$

This means that the *physical* vacuum state $\langle 0|$ could be thought of as made of M quarks above the *bare* vacuum $\langle \Omega|$

$$\langle 0| = \langle \Omega| \prod_{s=1}^{M} \left[\left(\phi^s_{\rm vac}\right)^+ b \right] = \langle \Omega| \prod_{s=1}^{M} \left[\sum_m \left(\phi^s_{\rm vac}\right)^*_m b_m \right] , \tag{9.60}$$

where the "single-particle" wave functions $\phi^S_{\rm vac}$ are solutions of the equation

$$\left(\phi^s_{\rm vac}\right)^+ \sum_\alpha \pi^{\rm vac}_\alpha \Gamma_\alpha = \left(\phi^s_{\rm vac}\right)^+ \varepsilon^s_{\rm vac} \, . \tag{9.61}$$

Using expression (9.60), we can compute the matrix element on the RHS of Eq. (9.48):

$$\langle 0| \left[b^+(0) \Gamma^\beta b(0) \right] |0\rangle_{\pi^{\rm vac}} = \sum_{s=1}^{M} \left(\phi^s_{\rm vac}\right)^+ \Gamma^\beta \phi^s_{\rm vac} \, . \tag{9.62}$$

Inserting this result into Eq. (9.48), we find

$$\pi^{\rm vac}_\alpha = \sum_\beta V_{\alpha\beta} \sum_{s=1}^{M} \left(\phi^s_{\rm vac}\right)^+ \Gamma^\beta \phi^s_{\rm vac} \, . \tag{9.63}$$

Now we see that Eqs. (9.61) and (9.63) are nothing else but the Hartree equations (2.10), (2.11) for the Hamiltonian (9.1). In the framework of these Hartree equations we have

$$E_{\rm vac} = \frac{1}{2} \sum_{s=1}^{M} \varepsilon^s_{\rm vac} \, . \tag{9.64}$$

Instead of the second quantized representation for the physical vacuum $|0\rangle$ we can use the language the of M-particle wave function written in terms of the Slater determinant

$$|0\rangle \to \frac{1}{\sqrt{M!}} \det_{1 \le s,i \le M} \left\| \left(\phi^s_{\rm vac}\right)_{n_i} \right\| \, . \tag{9.65}$$

Similar equations can be written for the baryon state $|B\rangle$ which is associated in the matrix elements (9.57), (9.58) with the limit $t \to +\infty$:

$$|B\rangle = \prod_{s=1}^{M+1} \left(\phi^s_{\rm bar} b^+ \right) |\Omega\rangle = \left\{ \prod_{s=1}^{M+1} \left[\sum_k \left(\phi^s_{\rm bar}\right)_k b^+_k \right] \right\} |\Omega\rangle \, , \tag{9.66}$$

$$\sum_\alpha \left(\pi^{\rm bar}_\alpha \Gamma_\alpha \right) \phi^s_{\rm bar} = \varepsilon^s_{\rm bar} \phi^s_{\rm bar} \, , \tag{9.67}$$

$$\pi^{\rm bar}_\alpha = \sum_\beta V_{\alpha\beta} \sum_{s=1}^{M+1} \left(\phi^s_{\rm bar}\right)^+ \Gamma^\beta \phi^s_{\rm bar} \, , \tag{9.68}$$

$$E_{\rm bar} = \frac{1}{2} \sum_{s=1}^{M+1} \varepsilon^s_{\rm bar} \, . \tag{9.69}$$

The state $|B\rangle$ corresponds to the Slater determinant

$$|B\rangle \to \frac{1}{\sqrt{(M+1)!}} \det_{1 \le s,i \le M+1} \left\| \left(\phi^s_{\rm bar}\right)_{n_i} \right\| \, . \tag{9.70}$$

We work with the pure 4-fermionic Hamiltonian (9.1) which has no quadratic piece. Therefore we have additional simplifications. Combining Eqs. (9.9), (9.61), (9.63), (9.64), (9.69), we find

$$E_{\rm vac} = L_{\rm bos}(\pi^{\rm vac}) = \sum_{s=1}^{M} \varepsilon^s_{\rm vac} = \frac{1}{2}\sum_{\alpha\beta} V_{\alpha\beta}\pi^{\rm vac}_\alpha \left(V^{-1}\right)_{\alpha\beta}\pi^{\rm vac}_\beta \tag{9.71}$$

and similarly for the baryon

$$E_{\rm bar} = L_{\rm bos}(\pi^{\rm bar}) = \sum_{s=1}^{M+1} \varepsilon^s_{\rm bar} = \frac{1}{2}\sum_{\alpha\beta} V_{\alpha\beta}\pi^{\rm bar}_\alpha \left(V^{-1}\right)_{\alpha\beta}\pi^{\rm bar}_\beta \,. \tag{9.72}$$

Now we can simplify expression (9.55)

$$W(g) = \ln\langle 0|\left[gb(t_1)\right]|B\rangle_{\pi^{(1)}} + t_1\left[\sum_{s=1}^{M+1}\varepsilon^s_{\rm bar} - \sum_{s=1}^{M}\varepsilon^s_{\rm vac}\right]$$
$$+ \int_{-\infty}^{t_1} dt\left[L_{\rm bos}(\pi^{(1)}) - L_{\rm bos}(\pi^{\rm bar})\right] + \int_{t_1}^{\infty} dt\left[L_{\rm bos}(\pi^{(1)}) - L_{\rm bos}(\pi^{\rm vac})\right] . \tag{9.73}$$

I. Time dependence

In the previous section we have shown how the static Hartree equations for the baryon and for the vacuum appear in the context of the calculation of the generating functional $W(g)$ for the baryon wave function. These static equations correspond to the asymptotic limit (9.56) of the field $\pi^{(1)}(t)$ at $t \to \pm\infty$.

Now we turn to the calculation of the time-dependent matrix elements (9.57), (9.58). These matrix elements are written in the Heisenberg representation with fixed states $\langle 0|$ and $|B\rangle$ but time-dependent operators $b(t)$ and $b^+(t)$. However, we can rewrite these matrix elements in terms of the Schrödinger representation. The Schrödinger representation is based on the time-dependent wave functions corresponding to the static Heisenberg wave functions $\langle 0|$ (9.60) and $|B\rangle$ (9.66):

$$\langle 0| \to \langle 0,t| = \langle\Omega|\prod_{s=1}^{M}\left[\tilde\phi^s_{\rm vac}(t)b\right] , \tag{9.74}$$

$$|B\rangle \to |B,t\rangle = \prod_{s=1}^{M+1}\left[\phi^s_{\rm bar}(t)b^+\right]|\Omega\rangle \,. \tag{9.75}$$

We use notation $\tilde\phi^s_{\rm vac}(t)$ instead of $[\phi^s_{\rm vac}(t)]^+$ because the Euclidean evolution is not unitary.

The time dependence of the wave functions $\tilde\phi^s_{\rm vac}(t)$ and $\phi^s_{\rm bar}(t)$ is controlled by the quadratic form $K(\pi^{(1)})$ (9.11) which defines the b, b^+ theory (9.31), (9.32). Obviously the role of the Hamiltonian in the quadratic form $K(\pi^{(1)})$ is played by

$$h(t) = \sum_\alpha \pi^{(1)}_\alpha(t)\Gamma_\alpha \,. \tag{9.76}$$

The corresponding evolution operator is

$$U(\tau_1,\tau_2) \overset{\tau_1\geq\tau_2}{=} T\exp\left[-\int_{\tau_2}^{\tau_1} dt\sum_\alpha \pi^{(1)}_\alpha(t)\Gamma_\alpha dt\right] , \tag{9.77}$$

$$U(\tau_1,\tau_2) \overset{\tau_1\leq\tau_2}{=} \left[U(\tau_2,\tau_1)\right]^{-1} . \tag{9.78}$$

This operator $U(\tau_1,\tau_2)$ controls the evolution of wave functions $\phi^s_{\rm vac}(t)$ and $\phi^s_{\rm bar}(t)$:

$$\tilde\phi^s_{\rm vac}(t) = \lim_{\tau\to+\infty}\left(\phi^s_{\rm vac}\right)^+ U(\tau,t)\exp\left(\tau\varepsilon^s_{\rm vac}\right) , \tag{9.79}$$

$$\phi^s_{\rm bar}(t) = \lim_{\tau\to-\infty} U(t,\tau)\phi^s_{\rm bar}\left(-\tau\varepsilon^s_{\rm bar}\right) . \tag{9.80}$$

On the RHS we have the time independent wave functions $\phi^s_{\rm vac}$ and $\phi^s_{\rm bar}$ which are determined by the Hartree equations (9.61), (9.63) for $\phi^s_{\rm vac}$ and (9.67), (9.68) for $\phi^s_{\rm bar}$.

For the matrix element (9.57), the reduction to the Schrödinger representation can be done as follows

$$\langle 0|\left[gb(t_1)\right]|B\rangle_{\pi^{(1)}} = \langle 0,t_1|\left(gb\right)|B,t_1\rangle_{\pi^{(1)}} = \langle\Omega|\left\{\prod_{r=1}^{M}\left[\tilde\phi^r_{\rm vac}(t_1)b\right]\right\}\left(gb\right)\left\{\prod_{s=1}^{M+1}\left[\phi^s_{\rm bar}(t_1)b^+\right]\right\}|\Omega\rangle . \tag{9.81}$$

Let us define

$$g(t) = gU(t_1,t) , \tag{9.82}$$

using the evolution operator (9.77). Next, let us introduce a special notation for the set of $M+1$ functions including $\phi^s_{\rm vac}(t)$ and $g(t)$

$$\tilde\phi^r_{{\rm vac},g}(t) = \begin{cases} \tilde\phi^r_{\rm vac}(t) & \text{if } 1\le r\le M , \\ g(t) & \text{if } r = M+1 . \end{cases} \tag{9.83}$$

Then we find from Eq. (9.81)

$$\langle 0|\left[gb(t_1)\right]|B\rangle_{\pi^{(1)}} = \langle\Omega|\left\{\prod_{r=1}^{M+1}\left[\tilde\phi^r_{{\rm vac},g}(t_1)b\right]\right\}\left\{\prod_{s=1}^{M+1}\left[\phi^s_{\rm bar}(t_1)b^+\right]\right\}|\Omega\rangle = \det_{1\le r,s\le M+1} D_{rs}(t_1) , \tag{9.84}$$

where we have introduced the notation

$$D_{rs}(t) = \tilde\phi^r_{{\rm vac},g}(t)\cdot\phi^s_{\rm bar}(t) . \tag{9.85}$$

Inserting Eqs. (9.79), (9.80) and using Eq. (9.78), we find

$$D_{rs}(t) = \lim_{\tau_1\to+\infty}\lim_{\tau_2\to-\infty}\exp\left(\tau_1\varepsilon^r_{\rm vac}-\tau_2\varepsilon^s_{\rm bar}\right)\left(\phi^s_{\rm vac}\right)^+ U(\tau_1,\tau_2)\phi^s_{\rm bar} . \tag{9.86}$$

We see that $D_{rs}(t)$ is t independent

$$\frac{d}{dt}D_{rs}(t) = 0 , \tag{9.87}$$

since the t evolution (9.79), (9.82) of $\tilde\phi^r_{{\rm vac},g}(t)$ compensates the t evolution (9.80) of $\phi^s_{\rm bar}(t)$.

By analogy with Eq. (9.81) we can write

$$\langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\beta b(t)\right]\right\}|B\rangle_{\pi^{(1)}} = \begin{cases} \langle 0,t|\left[g(t)b\right]\left(b^+\Gamma_\beta b\right)|B,t\rangle_{\pi^{(1)}} & \text{if } t<t_1, \\ \langle 0,t|\left(b^+\Gamma_\beta b\right)\left[g(t)b\right]|B,t\rangle_{\pi^{(1)}} & \text{if } t>t_1. \end{cases} \tag{9.88}$$

In the case $t<t_1$ this leads to

$$\begin{aligned}\langle 0|T\left\{\left[gb(t_1)\right]\left[b^+(t)\Gamma_\beta b(t)\right]\right\}|B\rangle_{\pi^{(1)}} &\overset{t<t_1}{=} \langle\Omega|\left\{\prod_{r=1}^{M+1}\left[\tilde\phi^r_{{\rm vac},g}(t_1)b\right]\right\}\left(b^+\Gamma_\beta b\right)\left\{\prod_{s=1}^{M+1}\left[\phi^s_{\rm bar}(t_1)b^+\right]\right\}|\Omega\rangle \\ &= \left(\det_{1\le r',s'\le M+1} D_{r's'}\right)\sum_{r,s=1}^{M+1}\left(D^{-1}\right)_{sr}\left[\tilde\phi^r_{{\rm vac},g}(t)\Gamma_\beta\phi^s_{\rm bar}(t)\right] .\end{aligned} \tag{9.89}$$

Inserting Eqs. (9.84) and (9.89) into Eq. (9.45), we obtain

$$\pi^{(1)}_\alpha(t) \overset{t<t_1}{=} \sum_\beta V_{\alpha\beta}\sum_{r,s=1}^{M+1}\left(D^{-1}\right)_{sr}\left[\tilde\phi^r_{{\rm vac},g}(t)\Gamma_\beta\phi^s_{\rm bar}(t)\right] . \tag{9.90}$$

According to Eq. (9.85) D_{rs} is a local (in t) function of $\phi_{\rm vac}(t)$, $\phi_{\rm bar}(t)$ and $g(t)$. Therefore Eq. (9.90) provides us a local expression for $\pi_\alpha(t)$ via $\phi_{\rm vac}(t)$, $\phi_{\rm bar}(t)$ and $g(t)$:

$$\pi^{(1)}_\alpha(t) \stackrel{t\le t_1}{=} \Pi^{(-)}_\alpha\left(\tilde\phi_{\rm vac}(t), \phi_{\rm bar}(t), g(t)\right) , \tag{9.91}$$

where

$$\Pi^{(-)}_\alpha\left(\tilde\phi_{\rm vac}, \phi_{\rm bar}, g\right) = \sum_\beta V_{\alpha\beta} \sum_{r,s=1}^{M+1} \left\{\left[\tilde\phi_{{\rm vac},g}\cdot\phi_{\rm bar}\right]^{-1}\right\}_{sr} \left[\tilde\phi^r_{{\rm vac},g}\Gamma_\beta\phi^s_{\rm bar}\right] . \tag{9.92}$$

The $\tilde\phi_{{\rm vac},g}$ is defined similarly to Eq. (9.83).

It is easy to see that the above argument can be repeated for the case $t > t_1$:

$$\pi^{(1)}_\alpha(t) \stackrel{t\ge t_1}{=} \Pi^{(+)}_\alpha\left(\tilde\phi_{\rm vac}(t), \phi_{\rm bar}(t), g(t)\right) \tag{9.93}$$

Inserting Eq. (9.84) into Eq. (9.73), we find

$$\begin{aligned} W(g) &= \ln \det_{1\le r,s\le M+1} D_{rs} + t_1\left[\sum_{s=1}^{M+1}\varepsilon^s_{\rm bar} - \sum_{s=1}^{M}\varepsilon^s_{\rm vac}\right] \\ &+ \int_{-\infty}^{t_1} dt\left[L_{\rm bos}(\pi^{(1)}) - L_{\rm bos}(\pi^{\rm bar})\right] + \int_{t_1}^{\infty} dt\left[L_{\rm bos}(\pi^{(1)}) - L_{\rm bos}(\pi^{\rm vac})\right] . \end{aligned} \tag{9.94}$$

According to Eq. (9.87) D_{rs} is independent of t. However, D_{rs} depends on t_1. This t_1 dependence is implicit. Indeed, if we shift t_1 in the original path integral representation,

$$t_1 \to t_1 + \Delta T , \tag{9.95}$$

then this will lead to the corresponding shift of the saddle point $\pi^{(1)}$

$$\pi^{(1)}(t) \to \pi^{(1)}(t - \Delta T) \tag{9.96}$$

and to the shift in the evolution operator

$$U(\tau_1, \tau_2) \to U(\tau_1 - \Delta T, \tau_2 - \Delta T) . \tag{9.97}$$

Therefore the matrix D_{rs} (9.86) will change as follows:

$$D_{rs} \to D_{rs} + \Delta T\left(\varepsilon^r_{\rm vac} - \varepsilon^s_{\rm bar}\right) . \tag{9.98}$$

We see that the ΔT shifts of t_1 (9.95) and D_{rs} (9.98) compensate each other in Eq. (9.94) so that the functional $W(g)$ is independent of t_1 as it should be. Using this t_1 independence, we can simplify our equations by choosing

$$t_1 = 0 . \tag{9.99}$$

Then Eq. (9.94) becomes

$$W(g) = \ln \det_{1\le r,s\le M+1} D_{rs} + \int_{-\infty}^{0} dt\left[L_{\rm bos}(\pi^{(1)}) - L_{\rm bos}(\pi^{\rm bar})\right] + \int_{0}^{\infty} dt\left[L_{\rm bos}(\pi^{(1)}) - L_{\rm bos}(\pi^{\rm vac})\right] . \tag{9.100}$$

In the rest of the calculation we will assume the choice (9.99).

Using the expression (9.9) for $L_{\rm bos}$ and the result (9.90) for $\pi^{(1)}$ at $t < 0$, we find

$$\begin{aligned} L_{\rm bos}(\pi^{(1)}) &= \sum_{\alpha\beta}\frac{1}{2}\pi^{(1)}_\alpha\left[\left(V^{-1}\right)_{\alpha\beta}\pi^{(1)}_\beta\right] = \sum_\alpha\frac{1}{2}\pi^{(1)}_\alpha\left\{\sum_{r,s=1}^{M+1}\left(D^{-1}\right)_{sr}\left[\tilde\phi^r_{{\rm vac},g}(t)\Gamma_\alpha\phi^s_{\rm bar}(t)\right]\right\} \\ &= \frac{1}{2}\left\{\sum_{r,s=1}^{M+1}\left(D^{-1}\right)_{sr}\tilde\phi^r_{{\rm vac},g}(t)\left(\sum_\alpha\pi^{(1)}_\alpha\Gamma_\alpha\right)\phi^s_{\rm bar}(t)\right\} . \end{aligned} \tag{9.101}$$

Here

$$\tilde{\phi}^r_{\mathrm{vac},g}(t)\left(\sum_\alpha \pi^{(1)}_\alpha \Gamma_\alpha\right)\phi^s_{\mathrm{bar}}(t) = \varepsilon^s_{\mathrm{bar}} D_{rs}\,. \tag{9.102}$$

Thus

$$L_{\mathrm{bos}}(\pi^{(1)}(t)) = \frac{1}{2}\sum_{r,s=1}^{M+1}\varepsilon^s_{\mathrm{bar}}\,. \tag{9.103}$$

We see that the RHS is t independent. Taking the limit $t\to-\infty$ and using Eq. (9.56), we obtain

$$L_{\mathrm{bos}}(\pi^{\mathrm{bar}}) = \frac{1}{2}\sum_{r,s=1}^{M+1}\varepsilon^s_{\mathrm{bar}}\,. \tag{9.104}$$

Combining Eqs. (9.103), (9.104), we find

$$\int_{-\infty}^{0} dt\left[L_{\mathrm{bos}}(\pi^{(1)}) - L_{\mathrm{bos}}(\pi^{\mathrm{bar}})\right] = 0\,. \tag{9.105}$$

In a similar way one can show that

$$\int_{0}^{\infty} dt\left[L_{\mathrm{bos}}(\pi^{(1)}) - L_{\mathrm{bos}}(\pi^{\mathrm{vac}})\right] = 0\,. \tag{9.106}$$

As a result, Eq. (9.94) can be simplified:

$$W_{\mathrm{p.i.}}(g) = \ln \det_{1\le r,s\le M+1} D_{rs}\,. \tag{9.107}$$

We use the label *p.i.* in order to emphasize that this result is obtained in the path integral approach.

J. Closed system of differential equations

The U evolution (9.77) of functions $\tilde{\phi}^s_{\mathrm{vac}}(t)$, $\phi^s_{\mathrm{bar}}(t)$, $g(t)$ described by Eqs. (9.79), (9.80), and (9.82) can be represented in terms of the differential equations

$$\frac{d}{dt}\tilde{\phi}^s_{\mathrm{vac}}(t) = \tilde{\phi}^s_{\mathrm{vac}}(t)\sum_\alpha \pi^{(1)}_\alpha(t)\Gamma_\alpha\,, \tag{9.108}$$

$$\frac{d}{dt}\phi^s_{\mathrm{bar}}(t) = -\sum_\alpha \pi^{(1)}_\alpha(t)\Gamma_\alpha\phi^s_{\mathrm{bar}}(t)\,, \tag{9.109}$$

$$\frac{d}{dt}g(t) = g(t)\sum_\alpha \pi^{(1)}_\alpha(t)\Gamma_\alpha\,. \tag{9.110}$$

Using expressions (9.91) and (9.93) for $\pi^{(1)}_\alpha$, we can rewrite these differential equations in the form

$$\frac{d}{dt}\tilde{\phi}^s_{\mathrm{vac}}(t) = \tilde{\phi}^s_{\mathrm{vac}}(t)\sum_\alpha \Pi^{(\pm)}_\alpha\left(\tilde{\phi}_{\mathrm{vac}}(t),\phi_{\mathrm{bar}}(t),g(t)\right)\Gamma_\alpha\,, \tag{9.111}$$

$$\frac{d}{dt}\phi^s_{\mathrm{bar}}(t) = -\sum_\alpha \Pi^{(\pm)}_\alpha\left(\tilde{\phi}_{\mathrm{vac}}(t),\phi_{\mathrm{bar}}(t),g(t)\right)\Gamma_\alpha\phi^s_{\mathrm{bar}}(t)\,, \tag{9.112}$$

$$\frac{d}{dt}g(t) = g(t)\sum_\alpha \Pi^{(\pm)}_\alpha\left(\tilde{\phi}_{\mathrm{vac}}(t),\phi_{\mathrm{bar}}(t),g(t)\right)\Gamma_\alpha\,, \tag{9.113}$$

where $\Pi^{(+)}_\alpha$ should be used for $t>0$ and $\Pi^{(-)}_\alpha$ for $t<0$ (we work with $t_1=0$). This is a closed coupled system of differential equations for functions $\phi^s_{\mathrm{vac}}(t), \phi^s_{\mathrm{bar}}(t), g(t)$. In addition we have the boundary conditions which follow from Eqs. (9.79), (9.80), and (9.82)

$$\tilde{\phi}^s_{\mathrm{vac}}(t) \overset{t\to\infty}{\longrightarrow} \exp\left(t\varepsilon^s_{\mathrm{vac}}\right)\left(\phi^s_{\mathrm{vac}}\right)^+\,, \tag{9.114}$$

$$\phi^s_{\rm bar}(t) \stackrel{t\to-\infty}{\longrightarrow} \exp\left(-t\varepsilon^s_{\rm bar}\right)\phi^s_{\rm bar}\,, \tag{9.115}$$

$$g(0) = g\,. \tag{9.116}$$

Note that the g dependence appears via the boundary condition (9.116). Solving the system of differential equations (9.111)–(9.113) with the boundary conditions (9.114)–(9.116), inserting the solution into Eq. (9.117) and using Eq. (9.85), we can compute the functional $W_{\rm p.i.}(g)$ (9.117):

$$W_{\rm p.i.}(g) = \ln \det_{1\le r,s\le M+1}\left[\tilde\phi^r_{{\rm vac},g}(0)\cdot\phi^s_{\rm bar}(0)\right]\,. \tag{9.117}$$

This construction solves the problem of the calculation of the functional $W_{\rm p.i.}(g)$. However, one can wonder how this representation for $W_{\rm p.i.}(g)$ derived in the framework of the path integral formalism is related to our results obtained in the Schrödinger approach. This question is studied in the next section.

X. EQUIVALENCE OF THE SCHRÖDINGER AND PATH INTEGRAL APPROACHES

A. Problem of equivalence

Our analysis of models with the nontrivial vacuum has resulted in two expressions for the generating functional $W(g)$. One expression (8.68) was derived from the Schrödinger equation using the operator formalism. The other result (9.117) for $W(g)$ was obtained in the path integral formalism. Certainly both approaches are equivalent and should lead to the same results. However, in order to see this equivalence explicitly some extra work is needed. In Sec. X B we demonstrate the equivalence for the simple case of models with the trivial vacuum and after that turn to the much more interesting models with the nontrivial vacuum.

B. Models with the trivial vacuum

Let us start from the simplest case when the physical vacuum coincides with the bare one. Then the set of occupied vacuum levels is empty

$$\left\{\tilde\phi^s_{\rm vac}(t)\right\} = \emptyset \tag{10.1}$$

and

$$E_{\rm vac} = 0\,. \tag{10.2}$$

The baryon is described by the Hartree equations (9.67)–(9.69) corresponding to one occupied level:

$$\sum_\alpha \left(\pi^{\rm bar}_\alpha \Gamma_\alpha\right)\phi_{\rm bar} = \varepsilon_{\rm bar}\phi_{\rm bar}\,, \tag{10.3}$$

$$\pi^{\rm bar}_\alpha = \sum_b V_{\alpha\beta}\left(\phi^+_{\rm bar}\Gamma_\beta\phi_{\rm bar}\right)\,, \tag{10.4}$$

$$E_{\rm bar} = \frac{1}{2}\varepsilon_{\rm bar}\,. \tag{10.5}$$

Since the physical vacuum coincides with the bare one, we have

$$\langle 0,t| = \langle 0|\,, \tag{10.6}$$

$$\langle 0|b = 0\,. \tag{10.7}$$

Therefore the matrix element (9.88) vanishes at $t > t_1$. According to Eq. (9.99) we work with $t_1 = 0$. Thus equation (9.88) results in

$$\langle 0|T\left\{[gb(0)]\left[b^+(t)\Gamma_\beta b(t)\right]\right\}|B\rangle_{\pi^{(1)}} = \theta(-t)\langle 0|\,[g(t)b]\left(b^+\Gamma_\beta b\right)|B,t\rangle_{\pi^{(1)}} = \theta(-t)\,[g(t)\Gamma_\beta\phi_{\rm bar}(t)]\,. \tag{10.8}$$

On the other hand, Eq. (9.84) gives

$$\langle 0|\,[gb(0)]\,|B\rangle_{\pi^{(1)}} = g(t)\phi_{\rm bar}(t) = {\rm const}\,, \tag{10.9}$$

which is t independent. Now we insert Eqs. (10.8) and (10.9) into Eq. (9.45):

$$\pi^{(1)}_\alpha(t) = \theta(-t)\sum_\beta V_{\alpha\beta}\frac{[g(t)\Gamma_\beta\phi_{\rm bar}(t)]}{[g(t)\phi_{\rm bar}(t)]}\,. \tag{10.10}$$

According to Eqs. (9.109) and (9.110) we have the differential equations

$$\frac{d}{dt}\phi_{\rm bar}(t) = -\sum_\alpha \pi^{(1)}_\alpha(t)\Gamma_\alpha\phi_{\rm bar}(t)\,, \tag{10.11}$$

$$\frac{d}{dt}g(t) = g(t)\sum_\alpha \pi^{(1)}_\alpha(t)\Gamma_\alpha \tag{10.12}$$

with the boundary conditions (9.115), (9.116)

$$\phi_{\rm bar}(t) \overset{t\to-\infty}{\longrightarrow} \exp\left(-t\varepsilon_{\rm bar}\right)\phi_{\rm bar}\,, \tag{10.13}$$

$$g(0) = g\,. \tag{10.14}$$

According to Eqs. (9.56), (10.3), and (10.12) the asymptotic behavior of $g(t)$ at $t \to -\infty$ is

$$g(t) \overset{t\to-\infty}{=} {\rm const}\,\phi^+_{\rm bar}\exp\left(t\varepsilon_{\rm bar}\right) \tag{10.15}$$

and the path integral result (9.117) for $W(g)$ becomes

$$W_{\rm p.i.}(g) = \ln\left[g(t)\phi_{\rm bar}(t)\right]\,. \tag{10.16}$$

Thus the path integral calculation of the functional $W_{\rm p.i.}(g)$ is reduced to solving differential equations (10.11), (10.12) with $\pi^{(1)}$ given by Eq. (10.10) and with boundary conditions (10.13), (10.14), (10.15).

According to Eq. (10.9) the RHS of Eq. (10.16) is t independent. Taking the limit $t \to -\infty$ and inserting the asymptotic behavior (10.15), we can determine the constant on the RHS of (10.15):

$$g(t) \overset{t\to-\infty}{=} e^{W_{\rm p.i.}(g)}\,\phi^+_{\rm bar}\exp\left(t\varepsilon_{\rm bar}\right)\,. \tag{10.17}$$

Let us show that this path integral result $W_{\rm p.i.}(g)$ is equivalent to the Schrödinger approach discussed in Sec. VI. Note that our path integral treatment dealt with the case $L_{mn} = 0$. In the Schrödinger approach this case was discussed in Sec. VI F.

In order to match the equations of Sec. VI F with the equations of the path integral approach one has to use the following "dictionary" connecting the two formalisms:

$$q(t) = g(t)\,, \tag{10.18}$$

$$p(t) = e^{-W_{\rm p.i.}(g)}\phi_{\rm bar}(t)\,, \tag{10.19}$$

$$\varepsilon_0 = \varepsilon_{\rm bar}\,, \tag{10.20}$$

$$I(g) = [g(t)\phi_{\rm bar}(t)]\,. \tag{10.21}$$

On the LHS we have the quantities appearing in Sec. VI F within the Schrödinger approach whereas the RHS is represented by the objects which were used in the path integral method.

Now it is easy to see that the boundary conditions (6.58), (6.59), and (6.60) correspond to Eqs. (10.18), (10.19), and (10.14), respectively. The Hamilton equations (6.5) are mirrors of Eqs. (10.11)–(10.12).

C. Variables $Q^{(L)}$, $P^{(L)}$

In the previous section we have demonstrated the equivalence of the Schrödinger and path integral approaches in the case of systems with the trivial vacuum. Now we want to consider the general case when the vacuum is nontrivial.

Our main lesson from the analysis of the systems with the trivial vacuum is that the equivalence between the two approaches can be established on the basis of the Hamiltonian description of trajectories used for the calculation of the functional $W(g)$.

In the case of the nontrivial vacuum we already have the representation for $W(g)$ in terms of trajectories obeying the differential equations (9.108)–(9.110). Now we want to rewrite these differential equations in the Hamiltonian form.

To this aim we must introduce the canonical variables. Note that the original differential equations (9.108)–(9.110) are written in terms of single-particle wave functions $\tilde{\phi}^s_{\rm vac}(t)$, $\phi^s_{\rm bar}(t)$. As we will see below, the Hamiltonian formalism is based on M- and $(M+1)$-particle wave functions including the Slater determinants associated with the states (9.74) and (9.75)

$$P^{(M+1)}_{m_1\ldots m_{M+1}}(t) = \underset{m_1\ldots m_{M+1}}{\rm Antisym} \prod_{s=1}^{M+1} [\phi^s_{\rm bar}(t)]_{m_s} = \frac{1}{(M+1)!} \det_{1\le r,s\le M+1} [\phi^r_{\rm bar}(t)]_{m_s} \ , \tag{10.22}$$

$$Q^{(M)}_{m_1\ldots m_M}(t) = \underset{m_1\ldots m_M}{\rm Antisym} \prod_{s=1}^{M} \left[\tilde{\phi}^s_{\rm vac}(t)\right]_{m_s} = \frac{1}{M!} \det_{1\le r,s\le M} \left[\tilde{\phi}^r_{\rm vac}(t)\right]_{m_s} \ . \tag{10.23}$$

We assume that the antisymmetrization operator Antisym is normalized by the condition

$$({\rm Antisym})^2 = {\rm Antisym}\,. \tag{10.24}$$

Assuming the choice (9.99) $t_1 = 0$, we deal with the M-particle states at $t > 0$ and with the $(M+1)$-particle states at $t < 0$. Therefore the $M+1$ particle "wave function" $P^{(M+1)}_{m_1\ldots m_{M+1}}$ is relevant for the description of the region $t < 0$ whereas the variable $Q^{(M)}_{m_1\ldots m_M}$ will be used at $t > 0$.

Now we want to introduce the canonically conjugate variables for $Q^{(M)}_{m_1\ldots m_M}$ and for $P^{(M+1)}_{m_1\ldots m_{M+1}}$:

$$P^{(M)}_{m_1\ldots m_M}(t) = (M+1) \sum_{m_{M+1}} g_{m_{M+1}}(t) \underset{m_1\ldots m_{M+1}}{\rm Antisym} \prod_{s=1}^{M+1} [\phi^s_{\rm bar}(t)]_{m_s} \ , \tag{10.25}$$

$$Q^{(M+1)}_{m_1\ldots m_{M+1}}(t) = \underset{m_1\ldots m_{M+1}}{\rm Antisym} \left\{ g_{m_{M+1}}(t) \prod_{s=1}^{M} \left[\tilde{\phi}^s_{\rm vac}(t)\right]_{m_s} \right\} = \frac{1}{(M+1)!} \det \left[\tilde{\phi}^r_{{\rm vac},g}(t)\right]_{m_s} \ . \tag{10.26}$$

On the RHS of the Eq. (10.26) we use notation $\tilde{\phi}^r_{{\rm vac},g}(t)$ (9.83).

D. Differential equations for $Q^{(L)}$, $P^{(L)}$

Now let us rewrite the differential equations (9.108)–(9.110) in terms of variables $Q^{(M)}$, $P^{(M)}$ for $t > 0$ and in terms of $Q^{(M+1)}$, $P^{(M+1)}$ for $t < 0$:

$$\frac{d}{dt} Q^{(L)}_{m_1\ldots m_L}(t) = \sum_\alpha \pi^{(1)}_\alpha(t) \left[\Gamma_\alpha \cdot Q^{(L)}\right]_{m_1\ldots m_L} \ , \tag{10.27}$$

$$\frac{d}{dt} P^{(L)}_{m_1\ldots m_L}(t) = -\sum_\alpha \pi^{(1)}_\alpha(t) \left[P^{(L)} \cdot \Gamma_\alpha\right]_{m_1\ldots m_L} \ , \tag{10.28}$$

where

$$L = \begin{cases} M & \text{if } t > 0\,, \\ M+1 & \text{if } t < 0\,, \end{cases} \tag{10.29}$$

and

$$\left[\Gamma_\alpha \cdot P^{(L)}\right]_{m_1 \ldots m_L} = \sum_n \left[(\Gamma_\alpha)_{m_1 n} P^{(L)}_{n m_2 \ldots m_L}(t) + (\Gamma_\alpha)_{m_2 n} P^{(L)}_{m_1 n \ldots m_L}(t) + \ldots + (\Gamma_\alpha)_{m_L n} P^{(L)}_{m_1 \ldots m_{L-1} n}(t) \right] , \quad (10.30)$$

$$\left[Q^{(L)} \cdot \Gamma_\alpha\right]_{m_1 \ldots m_L} = \sum_n \left[Q^{(L)}_{n m_2 \ldots m_L}(t) \, (\Gamma_\alpha)_{n m_1} + Q^{(L)}_{m_1 n \ldots m_L}(t) \, (\Gamma_\alpha)_{n m_2} + \ldots + Q^{(L)}_{m_1 \ldots m_{L-1} n}(t) \, (\Gamma_\alpha)_{n m_L} \right] . \quad (10.31)$$

The field $\pi^{(1)}_\alpha(t)$ is given by

$$\pi^{(1)}_\alpha(t) = \sum_\beta V_{\alpha\beta} \frac{\left[Q^{(L)}(t) \cdot \Gamma_\beta \cdot P^{(L)}(t)\right]}{\left[Q^{(L)}(t) \cdot P^{(L)}(t)\right]} , \quad (10.32)$$

where L is defined by Eq. (10.29) and

$$\left[Q^{(L)} \cdot P^{(L)}\right] = \sum_{m_1 \ldots m_L} Q^{(L)}_{m_1 \ldots m_L} P^{(L)}_{m_1 \ldots m_L} , \quad (10.33)$$

$$\left[Q^{(L)} \cdot \Gamma_\alpha \cdot P^{(L)}\right] = L \sum_{k n m_2 \ldots m_L} Q^{(L)}_{k m_2 \ldots m_L} \, (\Gamma_\alpha)_{kn} \, P^{(L)}_{n m_2 \ldots m_L} . \quad (10.34)$$

Obviously

$$\frac{d}{dt}\left[Q^{(L)}(t) \cdot P^{(L)}(t)\right] = 0 \quad (10.35)$$

and

$$(M+1) \sum_{m_1 \ldots m_{M+1}} \left[Q^{(M+1)}(0) \cdot P^{(M+1)}(0)\right] = \left[Q^{(M)}(0) \cdot P^{(M)}(0)\right] . \quad (10.36)$$

E. Hamiltonian interpretation of differential equations

Let us introduce the Poisson bracket

$$\{Q_{m_1 \ldots m_L}, P_{n_1 \ldots n_L}\} = \frac{1}{L!} \det_{jk} \left\| \delta_{m_j n_k} \right\| . \quad (10.37)$$

Then

$$\left\{ Q^{(L)}_{m_1 \ldots m_L}, \left[Q^{(L)} \cdot \Gamma_\beta \cdot P^{(L)}\right] \right\} = \left[Q^{(L)} \cdot \Gamma_\beta\right]_{m_1 \ldots m_L} , \quad (10.38)$$

$$\left\{ P^{(L)}_{m_1 \ldots m_L}, \left[Q^{(L)} \cdot \Gamma_\beta \cdot P^{(L)}\right] \right\} = - \left[\Gamma_\beta \cdot Q^{(L)}\right]_{m_1 \ldots m_L} . \quad (10.39)$$

Now Eqs. (10.27) and (10.28) take the form

$$\frac{d}{dt} Q^{(L)}_{m_1 m_2 \ldots m_L} = \sum_\alpha \pi_\alpha(t) \left\{ Q^{(L)}_{m_1 m_2 \ldots m_L}, [Q \cdot \Gamma_\alpha \cdot P] \right\} , \quad (10.40)$$

$$\frac{d}{dt} P^{(L)}_{m_1 m_2 \ldots m_L} = \sum_\alpha \pi_\alpha(t) \left\{ P^{(L)}_{m_1 m_2 \ldots m_L}, [Q \cdot \Gamma_\alpha \cdot P] \right\} . \quad (10.41)$$

Inserting $\pi^{(1)}_a$ from Eq. (10.32), we can rewrite these equations in the form

$$\frac{d}{dt} Q^{(L)}_{m_1 m_2 \ldots m_L} = \frac{\left\{ Q^{(L)}_{m_1 m_2 \ldots m_L}, H \right\}}{\left[Q^{(L)}(t) \cdot P^{(L)}(t)\right]} , \quad (10.42)$$

$$\frac{d}{dt} P^{(L)}_{m_1 m_2 \ldots m_L} = \frac{\left\{ P^{(L)}_{m_1 m_2 \ldots m_L}, H \right\}}{\left[Q^{(L)}(t) \cdot P^{(L)}(t)\right]} , \quad (10.43)$$

where the Hamiltonian H is

$$H = \frac{1}{2} \sum_{\alpha\beta} V_{\alpha\beta} \left[Q \cdot \Gamma_\alpha \cdot P \right] \left[Q \cdot \Gamma_\beta \cdot P \right] . \tag{10.44}$$

According to Eq. (10.35) the denominator $\left[Q^{(L)}(t) \cdot P^{(L)}(t) \right]$ is t independent along any solution so that this denominator can be treated as a constant.

F. Boundary conditions

Using Eq. (10.23), we can rewrite the boundary condition (9.114) in the form

$$Q^{(M)}_{m_1 \ldots m_M}(t) \overset{t \to \infty}{\longrightarrow} \frac{1}{M!} \exp \left(t \sum_{p=1}^{M} \varepsilon^p_{\text{vac}} \right) \det_{1 \le r,s \le M} (\phi^r_{\text{vac}})^*_{m_s} . \tag{10.45}$$

Similarly we find from Eqs. (9.115) and (10.22)

$$P^{(M+1)}_{m_1 \ldots m_{M+1}}(t) \overset{t \to -\infty}{\longrightarrow} \frac{1}{(M+1)!} \exp \left(-t \sum_{p=1}^{M+1} \varepsilon^p_{\text{bar}} \right) \det_{1 \le r,s \le M+1} (\phi^r_{\text{bar}})_{m_s} . \tag{10.46}$$

Comparing Eqs. (10.23) and (10.26), we find at $t = 0$

$$Q^{(M+1)}_{m_1 \ldots m_{M+1}}(0) = \underset{m_1 \ldots m_{M+1}}{\text{Antisym}} \left[Q^{(M)}_{m_1 \ldots m_M}(0) g_{m_{M+1}} \right] . \tag{10.47}$$

Similarly, the comparison of Eqs. (10.22) and (10.25) results in

$$P^{(M)}_{n_1 \ldots n_M}(0) = (M+1) \sum_{m_{M+1}} g_{m_{M+1}} P^{(M+1)}_{m_1 \ldots m_{M+1}}(0) . \tag{10.48}$$

G. Expression for $W(g)$

According to Eqs. (9.85), (10.22), and (10.26) we have

$$\det_{1 \le r,s \le M+1} D_{rs} = (M+1)! \left[Q^{(M+1)}(t) \cdot P^{(M+1)}(t) \right] . \tag{10.49}$$

Combining this with Eq. (9.107), we find

$$W_{\text{p.i.}}(g) = \ln \left\{ (M+1)! \left[Q^{(M+1)}(t) \cdot P^{(M+1)}(t) \right] \right\} . \tag{10.50}$$

Later we will check the equivalence of this result with the expression (8.68) obtained in the Schrödinger approach.

Note that the RHS of Eq. (10.50) is t independent according to Eq. (10.35). Using Eqs. (10.35) and (10.36), we can also write

$$W_{\text{p.i.}}(g) = \ln \left\{ M! \left[Q^{(M)}(t) \cdot P^{(M)}(t) \right] \right\} . \tag{10.51}$$

H. Asymptotic representation

The asymptotic behavior of $P^{(M)}_{m_1 \ldots m_M}(t)$ at $t \to +\infty$ is given by the expression

$$P^{(M)}_{m_1 \ldots m_M}(t) \overset{t \to \infty}{\longrightarrow} I(g) \exp \left(-t \sum_{p=1}^{M} \varepsilon^p_{\text{vac}} \right) \det_{1 \le r,s \le M} (\phi^r_{\text{vac}})_{m_s} \tag{10.52}$$

with some constant $I(g)$. According to Eq. (10.35) the combination $\left[Q^{(M)}(t)\cdot P^{(M)}(t)\right]$ is t independent so that it can be computed via its limit at $t\to+\infty$. Using Eqs. (10.45) and (10.52), we obtain

$$\left[Q^{(M)}(t)\cdot P^{(M)}(t)\right]=\lim_{t\to+\infty}\left[Q^{(M)}(t)\cdot P^{(M)}(t)\right]=I(g)\,. \tag{10.53}$$

Combining this with Eq. (10.36), we find

$$\left[Q^{(M+1)}(t)\cdot P^{(M+1)}(t)\right]=\frac{I(g)}{M+1}\,. \tag{10.54}$$

Taking the limit $t\to-\infty$ in this equation, we can fix the constant in the asymptotic expression

$$Q^{(M+1)}_{m_1\ldots m_{M+1}}(t)\overset{t\to-\infty}{\longrightarrow}\frac{I(g)}{M+1}\exp\left(t\sum_{p=1}^{M+1}\varepsilon^p_{\rm bar}\right)\det_{1\le r,s\le M+1}(\phi^r_{\rm bar})^*_{m_s}\,. \tag{10.55}$$

Inserting expression (10.53) into Eq. (10.51), we obtain

$$W_{\rm p.i.}(g)=\ln I(g)\,. \tag{10.56}$$

I. Comparison of path integral and Schrödinger results

We have computed the functional $W(g)$ using the operator (the Schrödinger) approach and the path integral method. The two results are given by Eqs. (8.68) and (10.56), respectively. Our path integral result (10.56) is more general since it deals with models where the physical ground state contains N_cM quarks with arbitrary M whereas the Schrödinger result (8.68) corresponds to $M=1$.

Comparing equations (8.68) and (10.56), we see that they give the same result for $W(g)$ if

$$I(g)=\left\{I_{\rm vac}\left(k(g^*)\right)\right\}^*I_{\rm bar}\left(Q(g)\right)\,. \tag{10.57}$$

The functional $I(g)$ arises in the path integral approach whereas the functionals $I_{\rm vac}\left(k(g^*)\right)$ and $I_{\rm bar}\left(Q(g)\right)$come from the Schrödinger approach. The aim of this section is to give an independent derivation of the identity (10.57) which directly establishes the equivalence of the path integral and Schrödinger methods.

The functionals appearing in Eq. (10.57) can be represented in terms of trajectories. In the path integral approach the trajectory representation for $I(g)$ arises via the saddle point method. The Schrödinger representation for $W(g)$ was formulated in Sec. VIII I in terms of trajectories $q^{\rm vac}_i(t)$, $p^{\rm vac}_i(t)$ for the vacuum sector and trajectories $Q^{\rm bar}_{ij}(t)$, $P^{\rm bar}_{ij}(t)$ for the baryon sector. Let us show that the connection between the trajectories of the Schrödinger approach and the trajectories $P^{(L)}(t),Q^{(L)}(t)$ (with $L=1,2$) of the path integral approach is given by equations

$$Q^{(1)}_i(t)=I^*_{\rm vac}\left[p^{\rm vac}_i(-t)\right]^*\,, \tag{10.58}$$

$$P^{(1)}_i(t)=I_{\rm bar}\left[q^{\rm vac}_i(-t)\right]^*\,, \tag{10.59}$$

$$Q^{(2)}_{ij}(t)=\frac{1}{2}I^*_{\rm vac}Q^{\rm bar}_{ij}(t)\,, \tag{10.60}$$

$$P^{(2)}_{ij}(t)=I_{\rm bar}P^{\rm bar}_{ij}(t)\,. \tag{10.61}$$

Remember that the trajectories $q^{\rm vac}_i(t)$, $p^{\rm vac}_i(t)$ were introduced in Sec. VIII I for $t<0$ whereas the trajectories $P^{(1)}(t)$, $Q^{(1)}(t)$ are defined at $t>0$. Note that the equations for the trajectories are invariant under the combination of the time reversal transformation $t\to-t$ with the complex conjugation and with the exchange $P^{(1)}\leftrightarrow Q^{(1)}$. Therefore functions $P^{(1)}(t)$, $Q^{(1)}(t)$ defined by Eqs. (10.58) and (10.59) automatically obey the equations of motion.

Next, the equation for trajectories are invariant under the rescaling transformations

$$Q^{(L)}\to a_LQ^{(L)}\,,\quad P^{(L)}\to b_LP^{(L)}\,. \tag{10.62}$$

Therefore various t independent factors appearing in Eqs. (10.58)–(10.61) do not violate the equations of motion.

Let us summarize. Eqs. (10.58)–(10.61) express the trajectories $Q^{(L)}(t)$, $P^{(L)}(t)$ of the path integral approach via the trajectories $q^{\rm vac}(t)$, $p^{\rm vac}(-t)$, $Q^{\rm bar}(t)$, $Q^{\rm bar}(t)$ of the Schrödinger approach. If the Schrödinger trajectories obey the

equations of motion associated with the Schrödinger approach then the trajectories $Q^{(L)}(t)$, $P^{(L)}(t)$ defined by Eqs. (10.58)–(10.61) satisfy the classical equations motion (10.42), (10.43) corresponding to the path integral approach.

The next step is to check that the boundary conditions (10.45)–(10.48) of the path integral approach also follow from the equations of the Schrödinger approach.

According to Eq. (8.55) we have

$$P^{(1)}(0) = I_{\rm bar} k^* \,. \tag{10.63}$$

Using Eqs. (8.57) and (10.58), we find

$$\frac{\partial W_{\rm vac}(k_n)}{\partial k_n} = \frac{1}{I_{\rm vac}} \left[Q_n^{(1)}(0) \right]^* \,. \tag{10.64}$$

Let us insert this equation into Eq. (8.67):

$$\frac{1}{I^*_{\rm vac}} \left[g_j Q_i^{(1)}(0) - g_i Q_j^{(1)}(0) \right] = Q_{ij} \,. \tag{10.65}$$

According to Eqs. (8.59) and (10.61) we have

$$I^*_{\rm vac} Q_{ij} = 2 Q_{ij}^{(2)}(0) \,. \tag{10.66}$$

Thus

$$g_j Q_i^{(1)}(0) - g_i Q_j^{(1)}(0) = 2 Q_{ij}^{(2)}(0) \,. \tag{10.67}$$

This is nothing else but the boundary condition (10.47) of the path integral approach.

Now let us check the boundary condition (10.48). We start from equation (8.66) and insert k from Eq. (10.63)

$$\sum_j \frac{\partial W_{\rm bar}(Q)}{\partial Q_{ij}} g_j = \frac{1}{I_{\rm bar}} P_i^{(1)}(0) \,. \tag{10.68}$$

Next we combine Eq. (8.63) with Eq. (10.61)

$$\frac{\partial W_{\rm bar}(Q)}{\partial Q_{ij}} = \frac{2}{I_{\rm bar}} P_{ij}^{(2)}(0) \,. \tag{10.69}$$

Inserting Eq. (10.69) into Eq. (10.68), we find

$$2 \sum_j P_{ij}^{(2)}(0) g_j = P_i^{(1)}(0) \,. \tag{10.70}$$

This shows that the boundary condition (10.48) of the path integral approach also holds.

Combining Eqs. (8.53)–(8.54) with Eqs. (10.59), (10.58), we find

$$P_n^{(1)}(t) \stackrel{t\to\infty}{=} \delta_{n0} I_{\rm bar} I_{\rm vac} \exp\left(-\varepsilon^1_{\rm vac} t\right) \,, \tag{10.71}$$

$$Q_n^{(1)}(t) \stackrel{t\to+\infty}{=} \delta_{n0} \exp\left(\varepsilon^1_{\rm vac} t\right) \,. \tag{10.72}$$

The last equation agrees with Eq. (10.45).

Next, according to Eqs. (8.60), (8.61), (10.60), (10.61) we have

$$Q_{ij}^{(2)}(t) \stackrel{t\to-\infty}{=} \frac{1}{2} \varepsilon_{ij} I^*_{\rm vac} I_{\rm bar} \exp\left[\left(\varepsilon^1_{\rm bar} + \varepsilon^2_{\rm bar}\right) t\right] \,, \tag{10.73}$$

$$P_{ij}^{(2)}(t) \stackrel{t\to-\infty}{=} \frac{1}{2} \varepsilon_{ij} \exp\left[-\left(\varepsilon^1_{\rm bar} + \varepsilon^2_{\rm bar}\right) t\right] \,. \tag{10.74}$$

The last equation agrees with Eq. (10.46).

Thus we have checked that Eqs. (10.58)–(10.61) connecting the trajectories of the path integral approach with the trajectories of the Schrödinger approach are compatible both with the equations of motion and with the boundary conditions.

Now we can turn to the proof of the equality of functionals $W_{\rm Sch}(g)$ and $W_{\rm p.i.}(g)$ associated with the Schrödinger and path integral methods. As was explained above, this proof reduces to the derivation of the identity (10.57). Using Eqs. (10.35), (10.71), (10.72) we can compute $I(g)$ (10.53), (10.53)

$$I(g) = \left[Q^{(1)}(t) \cdot P^{(1)}(t)\right] = 2\left[Q^{(2)}(t) \cdot P^{(2)}(t)\right] = I^*_{\rm vac} I_{\rm bar}\,. \tag{10.75}$$

This proves relation (10.57). Thus

$$W_{\rm Sch}(g) = W_{\rm p.i.}(g)\,. \tag{10.76}$$

The equivalence of the Schrödinger and path integral representations for $W(g)$ is checked.

J. Analyticity and time contour

The functional $\Phi_B(g)$ is holomorphic in g by construction (2.6). This analyticity is inherited in the large-N_c limit (2.7) by the functional $W(g)$, although this limit can be accompanied by the appearance of singularities in $W(g)$. The representation for $W(g)$ in terms of classical trajectories is compatible with the analyticity of $W(g)$. Indeed, the Hamiltonian (10.44) used in this representations is a holomorphic functions of coordinates and momenta. However, the configuration of trajectories used for the construction of $W(g)$ can lead to the singularities in $W(g)$. Examples of these singularities can be found in the analytical expressions for $W(g)$ in the rotator model analyzed in Sec. VII [see Eq. (7.74)] and in the naive quark model studied in Ref. [16].

The analyticity of $W(g)$ in g is closely related to the analyticity of the corresponding trajectories in t. In our path integral construction we used Euclidean time. However, the choice of the Euclidean time is not necessary. For example, in the Schrödinger approach the functional $W(g)$ can be represented in a form which requires no time and no trajectories. The Schrödinger representation is formulated in terms of the action which is defined without explicit usage of the time variable. But if one wants to represent this action in terms of trajectories, then one has to use some version of the time variable. Naively the choice of the time contour (Euclidean, Minkowski or more complicated) is not important. However, even in simple systems we can meet singularities restricting the freedom of the choice of the time contour. Our general representation for the functional $W(g)$ in terms of trajectories ignored this problem which requires an additional analysis in any particular model.

XI. CONCLUSIONS

A. Main results

The main aim of this work was to check the consistency of the assumptions used for the construction of the $1/N_c$ expansion for the baryon wave function in QCD. We have concentrated on those nonperturbative features which cannot be directly proved in QCD. On the other hand, these properties can be studied in detail in models preserving the basic structure of the $1/N_c$ expansion in QCD. Our check is quite successful: in the analysis of models we could explicitly demonstrate all basic features:

– exponential large-N_c behavior of the generating functional for the baryon wave function,

– universality of this behavior,

– factorization of the preexponential terms.

In the absence of the direct proof of these properties in QCD, this model analysis combined with the independent arguments of Refs. [16, 17] (based on the evolution equation, spin-flavor symmetry and soft-pion theorem) gives us a rather solid self-consistent picture of the baryon wave function in large-N_c QCD.

The main criterion for our choice of models was the possibility to perform the direct calculation of the $1/N_c$ expansion. In the simplest models we could obtain analytical results. In more general models the problem has been reduced to the solution of a coupled system of differential equations corresponding to the large-N_c effective classical dynamics.

B. Phenomenological applications

Although the construction of realistic models was not the aim of this work, our analysis still has a direct relation to phenomenology.

1) The practical applications of the $1/N_c$ expansion which can be of interest for phenomenology are based on the large-N_c contracted spin-flavor $SU(2N_f)$ symmetry. In the problem of the baryon wave function we have a rather nonstandard manifestation of this symmetry [17]. If we *assume* this realization of the spin-flavor symmetry, then we can derive interesting results like the connection between the distribution amplitudes of nucleon and Δ resonance. The aim of this paper was to *justify this assumption* by checking the structure of the $1/N_c$ expansion with a direct calculation.

2) The complete analysis of the effects related to the spin-flavor symmetry requires a thorough study of the next-to-next-to leading order (NNLO) of the $1/N_c$ expansion. In the current analysis we did not go so far. Most of the work was concentrated on the leading order (LO). We could also check the factorization properties appearing in the next-to-leading order (NLO). However, the subtle NNLO contributions crucial for the understanding of the spin-flavor symmetry effects remained beyond the scope of this paper. Nevertheless the successful LO and NLO check of the picture suggested in Refs. [16, 17] gives us additional evidence for the validity of this picture.

3) Although the results of this paper are presented in a simple quantum-mechanical form, they can be directly rewritten in field-theoretical terms relevant for the phenomenological models with four-fermionic interaction of the Nambu–Jona-Lasinio type.

C. Other aspects

The methods used in our analysis of large-N_c models are also interesting from the point of view of the relation between the large-N systems and classical dynamics. Our study of models has also revealed a connection between the equations for the generating functional $W(g)$ and such traditional equations of the many-body physics as the Hartree equation (including its time-dependent version) and RPA equation. Although the methods developed here cannot be applied directly to large-N_c QCD, they can be used in some special QCD problems like the diagonalization of the matrix of anomalous dimensions of the leading twist baryon operators [16].

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